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ABSTRACT

Intended to improve mathematics education at two-year colleges and other institutions offering lower division courses as well as to encourage more students to study mathematics, this publication presents the American Mathematical Association of Two-Year Colleges' (AMATYC's) standards for revitalizing the pre-calculus mathematics curriculum and stimulating changes in instructional methods. Following introductory sections, chapter 1 describes the goals and basic principles underlying the document, while chapter 2 presents standards for introductory college mathematics including seven standards related to intellectual development, seven related to curriculum content, and five related to pedagogy. This chapter also provides charts of guidelines for achieving the standards. Chapter 3 addresses issues of content and pedagogy related to the interpretation of the standards in the areas of mathematics foundation-building courses, technical programs, mathematics-intensive programs, liberal arts programs, and programs for prospective teachers. Chapter 4 reviews implications of the standards for faculty development and other departmental considerations; advising and placement; laboratory and learning center facilities; the use of technology; assessment of student outcomes; program evaluation; and articulation with high schools, other colleges and universities, and employers. Finally, chapter 5 covers implementation, including institutional recommendations, the role of professional organizations, proposed regional workshops, and the development of materials, while chapter 6 provides concluding remarks. (Contains 78 references.) (Sample math problems based on the standards are appended.) (KP)

CROSSROADS IN MATHEMATICS

ED 386 231

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CROSSROADS IN MATHEMATICS

Standards for Introductory College Mathematics before Calculus

Prepared by the Writing Team and Task Force of the Standards for Introductory College Mathematics Project, Don Cohen, Editor
September 1995

American Mathematical Association of Two-Year Colleges

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ENDORSEMENTS

The following professional organizations reviewed *Crossroads in Mathematics: Standards for Introductory College Mathematics Before Calculus* and endorse the philosophy and spirit of the document:

National Organizations

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Ohio Mathematics Association of Two-Year Colleges
Ohio Section of the MAA—Executive Committee
Pennsylvania Mathematics Association of Two-Year Colleges
Tennessee Mathematics Association of Two-Year Colleges
Utah Mathematical Association of Two-Year Colleges
Virginia Mathematical Association of Two-Year Colleges
Wisconsin Mathematical Association of Two-Year Colleges

Members of the Project Task Force

Project Director:
*Marilyn Mays
North Lake College
Irving, TX

Project Co-Director:
*Karen Sharp
Mott Community College
Flint, MI

Project Co-Director:
*Dale Ewen
Parkland College
Champaign, IL

Editor:
*Don Cohen
State University of New York
Cobleskill, NY

Darrell Abney
Maysville Community College
Maysville, KY

Geoffrey Akst
Manhattan Community College
New York, NY

Nancy S. Angle
Cerritos College
Norwalk, CA

Richelle Blair
Lakeland Community College
Mentor, OH

Linda H. Boyd
DeKalb College
Clarkston, GA

Max Cisneros, Jr.
Albuquerque Technical
Vocational Institute
Albuquerque, NM

Cheryl Cleaves
State Technical Institute at
Memphis
Memphis, TN

Betsy Darken
University of Tennessee at
Chattanooga
Chattanooga, TN

David Dudley
Phoenix College
Phoenix, AZ

Carol A. Edwards
St. Louis Community College,
Florissant Valley
St. Louis, MO

*Gregory D. Foley
Sam Houston State University
Huntsville, TX

Susan Forman
Mathematical Sciences
Education Board
Washington, DC

*Judith H. Hector
Walters State Community College
Morristown, TN

Margie Hobbs
State Technical Institute at Memphis
Memphis, TN

*Robert Kimball, Jr.
Wake Technical Community
College
Raleigh, NC

Edward D. Laughbaum
Columbus State Community
College
Columbus, OH

James R. C. Leitzel
University of Nebraska
Lincoln, NE

Myrna Manly (Advisor)
El Camino College
Torrence, CA

*Mercedes A. McGowen
William Rainey Harper
College
Palatine, IL

Dean Priest
Harding University
Searcy, AR

Bobby Righi
Seattle Community College
Seattle, WA

*Stephen B. Rodi
Austin Community College
Austin, TX

*William N. Thomas, Jr.
The University of Toledo
Community and Technical
College, Scott Park
Toledo, OH

Sam White
Jefferson State Community
College
Birmingham, AL

Susan S. Wood
J. Sargeant Reynolds
Community College
Richmond, VA

*Kathie Yoder
L.A. Pierce Community College
Woodland Hills, CA

Consultants

Sol Garfunkel
COMAP, Inc.
Lexington, MA

Harvey Keynes
University of Minnesota
Minneapolis, MN

Uri Treisman
The University of Texas
Austin, TX

*** Denotes Members of the Writing Team**

Members of the Planning Group Convened by the Mathematical Sciences Education Board

Max Cisneros, Jr.
Albuquerque Technical & Vocational Institute

Michael H. Clapp
MSEB Associate Executive Director

Cheryl Cleaves
State Technical Institute at Memphis

Ron Davis
DeKalb College

Carol A. Edwards
St. Louis Community College at Florissant Valley

Dale Ewen
Parkland College

Susan Forman
MSEB, Director of College and University Programs

Wanda Garner
Cabrillo College

Marilyn Mays
North Lake College

Calvin C. Moore
Member MSEB Executive Committee

David Pierce
President, American Association of Community Colleges

John Romo
Vice President for Academic Affairs
Santa Barbara Community College

Karen Sharp
Mott Community College

Ray C. Shiflett
MSEB Executive Director

Lynn A. Steen
Member MSEB Executive Board

Julian Weissglass
MSEB Consultant

Members of the National Steering Committee

John S. Bradley
American Mathematical Society

Linda H. Boyd
DeKalb College

Cheryl Cleaves
State Technical Institute at Memphis

Max Cisneros, Jr.
Albuquerque Technical and Vocational Institute

Don Cohen
SUNY at Cobleskill

Dale Ewen
Parkland College

Susan Forman
MSEB, Director of College and University Programs

Wanda Garner
Cabrillo College

Margie Hobbs
State Technical Institute at Memphis

Julie A. Keener
Central Oregon Community College

Genevieve Knight
Coppin State College

James R. C. Leitzel
University of Nebraska

Robert Malena
Community College of Allegheny County

Marilyn Mays
North Lake College

Dean Priest
Harding University

Janet P. Ray
Seattle Central Community College

Stephen B. Rodi
Austin Community College

Karen Sharp
Mott Community College

Christine Thomas
Benjamin Bannekar H.S.

Uri Treisman
University of Texas, Austin

Julian Weissglass
University of California, Santa Barbara

Assisting with the Steering Committee Meeting Were:

Richelle Blair
Ohio Board of Regents

Marilyn L. Hala
National Council of Teachers of Mathematics

Robert Kimball, Jr.
Wake Technical Community College

Ray C. Shiflett
California State Polytechnic University at Pomona

Members of the Advisory Panel

Paul M. Eakin, Jr.
University of Kentucky
Lexington, KY

Sheldon P. Gordon
Suffolk Community College
Selden, NY

Genevieve Knight
Coppin State College
Baltimore, MD

Mary Lindquist
Columbus College
Columbus, GA

Alan C. Tucker
SUNY at Stony Brook
Stony Brook, NY

Zalman P. Usiskin
University of Chicago
Chicago, IL

Bert K. Waits
Ohio State University
Columbus, OH

Ann E. Watkins
California State University, Northridge
Northridge, CA

CONTENTS

PREFACE ix

FOREWORD xii

CHAPTER 1 1
INTRODUCTION
- Introductory College Mathematics
- The Need for *Crossroads in Mathematics: Standards for Introductory College Mathematics Before Calculus*
- Basic Principles
- Summary

CHAPTER 2 7
STANDARDS FOR INTRODUCTORY COLLEGE MATHEMATICS
- Framework for Mathematics Standards
- Standards for Intellectual Development
- Standards for Content
- Standards for Pedagogy
- Guidelines for Achieving the Standards
- Summary

CHAPTER 3 21
INTERPRETING THE STANDARDS
- The Foundation
- Technical Programs
- Mathematics-Intensive Programs
- Liberal Arts Programs
- Programs for Prospective Teachers

CHAPTER 4 49
IMPLICATIONS
- Faculty Development and Departmental Considerations
- Advising and Placement
- Laboratory and Learning Center Facilities
- Technology
- Assessment of Student Outcomes
- Program Evaluation
- Articulation with High Schools, with Other Colleges and Universities, and with Employers
- Summary

CHAPTER 5 61
IMPLEMENTATION
- Institutional Recommendations
- Professional Organizations
- Other Recommendations
- Proposed Regional Workshops
- Development of Materials
- Summary

CHAPTER 6 67
LOOKING TO THE FUTURE

APPENDIX: 71
ILLUSTRATIVE EXAMPLES

REFERENCES 85

PREFACE

Mathematics programs at two-year and four-year colleges as well as at many universities serve students from diverse personal and academic backgrounds who begin their postsecondary educations with a wide variety of educational goals and personal aspirations. In addition to serving students who are prepared to study calculus, these mathematics programs must accommodate students who intend to study calculus but enter college unprepared to do so. They must also serve students who do not intend to study calculus. The career aspirations of some of these students are such that requirements for graduation and for job placement can be satisfied through the study of mathematics below the level of calculus. Mathematics taught at this level in two-year colleges and in the lower division of four-year colleges and universities is referred to in this document as "introductory college mathematics." This phrase will be used to include college algebra, trigonometry, introductory statistics, finite mathematics, and precalculus, as well as all courses presently characterized as developmental mathematics. Noncalculus-based mathematics courses for technical or occupational programs and mathematics courses for elementary teachers and those for liberal arts majors are also considered part of introductory college mathematics for the purposes of this document.

Introductory college mathematics constitutes a large percentage of the offerings at postsecondary institutions. The survey done in 1990 by the Conference Board of the Mathematical Sciences (Albers, Loftsgaarden, Rung, & Watkins, 1992) revealed the following data concerning students studying introductory college mathematics (enrollment in computer science is not considered in these data):

Of 1,295,000 students studying mathematics in two-year college mathematics departments,

- 724,000 (56%) were studying at the remedial level
- 245,000 (19%) were studying precalculus
- 17,000 were studying technical mathematics with no calculus prerequisite
- 35,000 were studying mathematics for liberal arts
- 9,000 were studying mathematics for elementary teachers

In addition, approximately another 126,000 students were studying introductory mathematics in two-year colleges in departments other than mathematics departments.

In four-year college and university mathematics departments,

- 261,000 (15% of the mathematics enrollment) were studying at the remedial level
- 593,000 (34% of the mathematics enrollment) were studying precalculus

These statistics indicate that introductory mathematics courses serve the needs of more than half the students studying mathematics in college.

The need for change in mathematics education has been documented in several national reports issued in the past decade, and significant change has begun at several levels. The *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 1989) presents comprehensive recommendations for innovative approaches to curriculum and pedagogy for kindergarten through twelfth grade; *A Curriculum in Flux* (Davis, 1989) makes specific recommendations for the curriculum at two-year colleges; *Reshaping College Mathematics* (Steen, 1989) outlines a proposed undergraduate curriculum; *Moving Beyond Myths* (National Research Council [NRC], 1991) calls for dramatic changes to "revitalize" undergraduate education; and *Everybody Counts* (NRC, 1989) makes specific recommendations for changes in mathematics programs from kindergarten through graduate school. Furthermore, calculus instruction has been reformed at many institutions [see Crocker (1990) for a description of the development of calculus reform, Ross (1994) for a description of recent reform initiatives, and Tucker and Leitzel (1994) for an assessment of calculus reform].

Until now no group has attempted to establish standards for mathematics programs that specifically address the needs of college students who plan to pursue careers that do not depend on knowledge of calculus or upper-division

mathematics, or those students who need calculus but enter college unprepared for mathematics at that level. Almost all postsecondary institutions offer introductory mathematics courses, but in two-year colleges these courses constitute over 80 percent of the offerings (Albers et al., 1992). The American Mathematical Association of Two-Year Colleges (AMATYC) is the organization whose primary mission includes the development and implementation of curricular, pedagogical, assessment, and professional standards for mathematics in the first two years of college. In this document, AMATYC, with assistance from other professional mathematics organizations, has undertaken the challenge of setting standards for curriculum and pedagogy in introductory college mathematics.

Building upon the reform efforts cited above this document presents standards that are designed for adult students, many of whom are underprepared for the study of college-level mathematics. The purpose of *Crossroads in Mathematics: Standards for Introductory College Mathematics Before Calculus* is to address the special circumstances of, establish standards for, and make recommendations about two-year college and lower-division mathematics programs below the level of calculus.

Three sets of standards for introductory college mathematics are defined in Chapter 2.

The **Standards for Intellectual Development** address desired modes of student thinking and represent goals for student outcomes. They include

- Problem Solving
- Modeling
- Reasoning
- Connecting with Other Disciplines
- Communicating
- Using Technology
- Developing Mathematical Power

The **Standards for Content** provide guidelines for the selection of content that will be taught at the introductory level. They include

- Number Sense
- Symbolism and Algebra
- Geometry
- Function
- Discrete Mathematics
- Probability and Statistics
- Deductive Proof

The **Standards for Pedagogy** recommend the use of instructional strategies that provide for student activity and interaction and for student-constructed knowledge. They include

- Teaching with Technology
- Interactive and Collaborative Learning
- Connecting with Other Experiences
- Multiple Approaches
- Experiencing Mathematics

The chapters that follow interpret the standards in various program areas, discuss the implications of the standards in several areas of mathematics education, and provide the design for establishing a nationwide effort to disseminate and implement the standards. Illustrative examples of problems aimed at capturing the vision and spirit of the standards appear in the Appendix.

The standards included in this document reflect many of the same principles found in school reform [for example, see NCTM (1989)] and calculus reform [see Crocker (1990), Ross (1994), and Tucker and Leitzel (1994)]. However, they differ in some respects and focus on the needs and experiences of college students studying introductory mathematics. In particular,

• **The Foundation** includes topics traditionally taught in "developmental mathematics" but also brings in additional topics that all students must understand and be able to use. Courses at this level should not simply be repeats of those offered in high school. Arithmetic, algebra, geometry, discrete mathematics, probability, and statistical concepts should be integrated into an in-depth applications-driven curriculum. The goal of this curriculum is to expand the educational and career options for all underprepared students.

• **Technical Programs** place strong emphasis on mathematics in the context of real applications. The mathematics involved is beyond the level of sophistication experienced in the Foundation. Mathematics faculty, in cooperation with their colleagues in technical areas or with outside practitioners, should select content that prepares students for

the immediate needs of employment. However, at the same time, students should learn to appreciate mathematics and to use mathematics to solve problems in a variety of fields so that they will be able to adapt to change in their career and educational goals.

• **The Mathematics-Intensive, Liberal Arts, and Prospective Teachers Programs** place heavy emphasis on using technology, developing general strategies for solving real-world problems, and actively involving students in the learning process. Students in each of these programs are either pursuing bachelor's degrees or intending to pursue bachelor's degrees after completing their associate's degrees. Introductory college mathematics is intended to provide the needed prerequisite knowledge for further study of mathematics or for courses in other disciplines that require a knowledge of mathematics at the introductory level. At the same time, liberal arts majors and prospective elementary school teachers should gain an appreciation for the roles that mathematics will play in their education, in their careers, and in their personal lives.

Cover Design

The cover art, created by graphics artist Karen Meyer Rappaport, depicts two crossing roads. These roads serve as a metaphor for two intersecting national trends: a growing societal need for a citizenry with a sophisticated level of mathematical preparation and an increasing number of academically underprepared students seeking entrance to postsecondary education. The cube situated at the intersection represents the three dimensions of the standards – intellectual development, content, and pedagogy – that are intended to guide mathematics faculty along the way to reform.

Vignettes and Margin Notes

The vignettes that appear throughout the document are true stories that help give a human face to the issues the report raises. The margin notes are of two types: excerpts that emphasize key ideas, and quotes from experienced mathematics educators and students that support the views expressed in the document. The quotes were obtained from speeches, published materials, position papers, document reviews, and student reports. The unidentified quotes may be attributed to the Task Force.

Acknowledgments

Each of the groups listed in the front matter played an important role in the planning and development of this document. The work of the Planning Group led to the development of the project proposal. The National Steering Committee developed principles to guide the writing of the document. The Task Force, aided by the consultants, began development of the document at a week-long writing session in Memphis in June 1993. The Task Force chose some of its members to serve as a writing team to receive feedback on the draft and revise the document accordingly. The Advisory Panel members reviewed the final draft and offered suggestions for final revisions. We wish to thank the numerous reviewers whose perspectives were considered at each stage of development. A special thank you is extended to Daniel Alexander, Michael Davidson, Sheldon Gordon, John Jenkins, and Jack Rotman, who submitted position papers, quotes from which appear as margin notes. The Task Force wishes to thank Allyn Jackson for helping edit the final draft, and artist Karen Meyer Rappaport for designing the cover. We thank Addison-Wesley Publishing Company for contributing the artwork and Prentice Hall, Inc. for donating the page designs and copy editing.

This document is intended to stimulate faculty to reform introductory college mathematics before calculus. These standards are not meant to be the "final word." Rather, they are a starting point for your actions.

Don Cohen
Editor

FOREWORD

Crossroads in Mathematics: Standards for Introductory College Mathematics Before Calculus has two major goals: to improve mathematics education at two-year colleges and at the lower division of four-year colleges and universities and to encourage more students to study mathematics. The document presents standards that are intended to revitalize the mathematics curriculum preceding calculus and to stimulate changes in instructional methods so that students will be engaged as active learners in worthwhile mathematical tasks. In addition, the implications of these changes in such areas as faculty development and student assessment are discussed. Preparation of these standards has been guided by the principle that faculty must help their students think critically, learn how to learn, and find motivation for the study of mathematics in appreciation of its power and usefulness.

This document represents a major effort of the American Mathematical Association of Two-Year Colleges (AMATYC), assisted by representatives of many other national mathematics education organizations. AMATYC's previous efforts to improve mathematics education at the two-year college level have taken the form of development of policy statements and guidelines. The most notable recent efforts have been *Guidelines for the Academic Preparation of Mathematics Faculty at Two-Year Colleges* (AMATYC, 1992) and *Guidelines for Mathematics Departments at Two-Year Colleges* (AMATYC, 1993). AMATYC was inspired toward additional efforts in education reform when in August 1991 the Mathematical Sciences Education Board (MSEB) convened a small planning group at the National Academy of Sciences to discuss two-year college mathematics education. Several members of the AMATYC leadership met with representatives from MSEB and others interested in two-year college mathematics education. This meeting and the subsequent efforts of the representatives of MSEB served to focus and launch the initiative.

The work of the Planning Group led to the development of a multiple-phase proposal designed to address curriculum and pedagogy reform initiatives at two-year colleges. The Exxon Education Foundation provided funds to assist with the initiative. AMATYC also committed funds while at the same time seeking additional support from other sources. Subsequently, the focus of the initiative was broadened to include all lower-division mathematics education below calculus, and substantial funding was received from the National Science Foundation (NSF). Marilyn Mays was designated Principal Investigator, with Karen Sharp and Dale Ewen as Co-Principal Investigators. A steering committee was formed with representatives from AMATYC, the American Mathematical Society (AMS), the Mathematical Association of America (MAA), MSEB, the National Association for Developmental Education (NADE), and the National Council of Teachers of Mathematics (NCTM). This group met in February 1993, in Washington, D.C., to plan the process for the development of a standards document.

The Steering Committee determined that the following principles should guide the deliberations of the Task Force:

- Fundamental changes are occurring in mathematics education as a result of the impact of several national documents, calculus reform efforts, technological advances, and research into how students learn.
- Colleges and universities must prepare an increasingly diverse group of students for further study and the world of work.
- While two-year college and lower-division mathematics students are preparing for a multitude of future occupations, there exists a common core of mathematical experiences, viewpoints, concepts, and skills that should be learned by all students.
- The manner in which students learn is inseparable from the content.
- Research regarding how students learn mandates development of new pedagogical methods and implementation of proven teaching techniques.
- The impact of technology both as a mode of instructional delivery and as a mathematical tool requires a redefinition of the mathematics curriculum.
- The demands of the workplace require that all students become empowered citizens capable of critical thinking.

The Steering Committee appointed a Task Force to develop a standards document. The Task Force was asked to formulate recommendations for a foundation program that would serve all students lacking preparation for college-level mathematics courses and to examine the special needs of students in technical, mathematics-intensive, liberal arts, and teacher preparation programs.

Before meeting, members of the Task Force reviewed a large collection of documents and articles on mathematics education reform and wrote position statements detailing their visions of the curricular and pedagogical reforms needed. These statements were distributed to the Task Force prior to the meeting.

The Task Force shaped a common vision for its work at its June 1993 meeting at State Technical Institute at Memphis. They also developed sections of the first draft of this document, originally titled *Standards for Curriculum and Pedagogical Reform in Two-Year College and Lower Division Mathematics.* Those sections were refined in subsequent weeks by the participants. Editor Don Cohen compiled them into a common format and produced a noncirculating draft which went back to the Task Force, to the National Steering Committee, and to a few reviewers and leaders of the mathematics education community. Then the document was further refined and published as a circulating draft. This draft was mailed to all AMATYC members and has been widely distributed outside of the two-year college community. Hearings were held at several national, regional, and state conferences. Reviews were solicited.

A writing team selected from the Task Force met in Dallas, February 24-27, 1994, to consider all of the comments made on the first circulating draft and to start the revision process. A second circulating draft was prepared and sent to the Task Force and other selected reviewers on July 15, 1994. A preliminary final draft was published in October 1994. It was distributed and discussed at several regional and national conferences including the November 1994 AMATYC conference in Tulsa, the January 1995 AMS-MAA joint mathematics meetings in San Francisco, the February 1995 NADE conference in Chicago, and the April 1995 NCTM conference in Boston. *Crossroads in Mathematics* is based on the many comments and reviews that were received.

The members of the Task Force, along with the names of members of the Writing Team, Planning Group, Steering Committee, and Advisory Panel are listed in the front matter.

We are grateful to the National Science Foundation and the Exxon Education Foundation, as well as to all who contributed to the development of this document.

Marilyn Mays
Project Director
AMATYC President
(1993-95)

Karen Sharp
Project Co-Director
AMATYC President
(1991-93)

Dale Ewen
Project Co-Director
AMATYC President
(1989-91)

CHAPTER 1:
Introduction

The ultimate goals of this document are to improve mathematics education and to encourage more students to study mathematics.

BEST COPY AVAILABLE

One day, about one-third of the way through the spring semester, an experienced mathematics faculty member walked into her department chair's office with a worried look. She had volunteered to teach an experimental section of Intermediate Algebra using a textbook that emphasized real problem solving, group projects, and technology. Algebra techniques were introduced as they were needed to solve problems. For example, while there was considerable work that involved linear, quadratic, exponential, and logarithmic functions, as well as systems of linear equations, there was little, if any, work on such traditional algebra topics as factoring, theory of equations, and the solution of rational and radical equations. She explained to the department chair that it was really refreshing to teach the new material using innovative instructional methods. Furthermore, her students, who were hesitant at the beginning of the semester to get involved in group projects, were also beginning to enjoy the new approach to studying mathematics. So, why the worried look?

When she had volunteered to use what she considered an exciting approach to learning algebra, she thought that she understood the underlying principles of the much discussed reform in mathematics education. She wanted to get involved. Now she wasn't sure that she was doing the right thing. Her course did not include many of the topics that had been the mainstay of intermediate algebra. She was worried that she was not adequately preparing her students for the study of higher levels of mathematics either at her college or at transfer institutions. ■

If you were the department chair, how would you have responded to your faculty member's concerns?

"The reform of school mathematics provides the educational community with possibilities for addressing the needs of an increasingly diverse student population. As they develop policy, research, and practice, educators will need to combine concerns for both equity and reform. If they fail to do so, students who do not come from dominant groups may, once again, be denied full participation."

Walter Secada, in
NCRMSE Research Review,
Fall 1994, p. 1.

THE NEED FOR *CROSSROADS IN MATHEMATICS: STANDARDS FOR INTRODUCTORY COLLEGE MATHEMATICS BEFORE CALCULUS*

Higher education is situated at the intersection of two major crossroads: A growing societal need exists for a well-educated citizenry and for a workforce adequately prepared in the areas of mathematics, science, engineering, and technology while, at the same time, increasing numbers of academically underprepared students are seeking entrance to postsecondary education.

Mathematics is a vibrant and growing discipline. New mathematics is continually being developed and is being used in more ways by more people than ever before. In fact, the rate of growth in mathematically based occupations is about twice that for all other occupations (NRC, 1990). Yet, an alarming situation now exists in postsecondary mathematics education. More students are entering the mathematics "pipeline" at a point below the level of calculus, but there has been no significant gain in the percentages of college students studying calculus (Albers et al., 1992). The purpose of *Crossroads in Mathematics* is to address the special circumstances of, establish standards for, and make recommendations about introductory college mathematics. The ultimate goals of this document are to improve mathematics education and encourage more students to study mathematics.

The students addressed in this document are seeking Associate of Arts (AA), Associate of Science (AS), Associate of Applied Science (AAS), and bachelor's

degrees. Some are traditional full-time students who are recent high school graduates. Others, particularly those at two-year colleges, are from widely diverse populations and fall into one or more of the following categories. They

- are older,
- work a full- or part-time job while attending college,
- manage a household,
- are returning to college after an interruption in their education of several years,
- intend to enter the work force after obtaining an associate degree,
- intend to work toward a bachelor's degree either at a transfer institution or in the upper division of their present four-year college or university,
- are studying for a degree as a part-time student,
- have English as a second language,
- need formal developmental work in a variety of disciplines and in study skills,
- have no family history in postsecondary education, or
- have disabilities that require special accommodations.

All of these characteristics dramatically affect introductory college mathematics instruction.

"... declining educational standards eliminate some applicants from consideration. We estimate that 30 to 40% have limited mathematics and science fundamentals, and that concerns us."

Jim Palarski, Marriott Hotels.
In *Mathematics Education, Wellspring of U.S. Industrial Strength*, 1989, p2.

BASIC PRINCIPLES

The following principles form the philosophical underpinnings of this document:

- *All students should grow in their knowledge of mathematics while attending college.* College students who are not prepared for college-level mathematics upon entering college can obtain the knowledge necessary by studying the Foundation (as described in Chapter 3). These students, along with others who enter college prepared for college-level mathematics, will continue to study mathematics to reach the level of sophistication required for their educational, career, and life goals.

- *The mathematics that students study should be meaningful and relevant.* Basic skills, general principles, algorithms, and problem-solving strategies should be introduced to the students in the context of real, understandable problem-solving situations so that students gain an appreciation for mathematics as a discipline, are able to use it as a base for further study, and can transfer this knowledge to problem-solving situations at work or in everyday life. Intuitive justifications for mathematical principles and procedures should be emphasized.

"If there is danger to the status of mathematics, it does not arise from overemphasis of its 'reasonableness.' It comes from the deadly overemphasis on routine 'techniques,' and the unwholesome neglect of its 'reasonableness' and of its 'relevance' to the real world."

Moses Richardson,
American Mathematical Monthly, 1942, p. 505.

- *Mathematics must be taught as a laboratory discipline.* Effective mathematics instruction should involve active student participation. In-depth projects employing genuine data should be used to promote student learning through guided hands-on investigations.

- *The use of technology is an essential part of an up-to-date curriculum.* Faculty and students will make effective use of appropriate technology. The technology available to students should include, but not be limited to, that used by practitioners in the field. Faculty should take advantage of software and graphing calculators that are designed specifically as teaching and learning tools. The technology must have graphics, computer algebra, spreadsheet, interactive geometry, and statistical capabilities.

- *Students will acquire mathematics through a carefully balanced educational program that emphasizes the content and instructional strategies recommended in the standards along with the viable components of traditional instruction.* These standards emphasize problem solving, the use of technology, intuitive understanding, and collaborative learning strategies. Skill acquisition and mathematical abstraction and rigor, however, are still critical components of mathematics education. Furthermore, direct whole-class instruction (lecturing, questioning, and discussion) is a viable option when working with highly structured content (Secada, 1992).

- *Introductory college mathematics should significantly increase students' options in educational and career choices.* When students master the content of introductory college mathematics, they will have the problem-solving skills that are required in many disciplines and careers.

- *Increased participation by all students in mathematics and in careers using mathematics is a critical goal in our heterogeneous society.* Mathematics instruction must reach out to all students: women, minorities, and others who have traditionally been underrepresented in the discipline, as well as students with learning difficulties, differing learning styles, disabilities, and language and socialization difficulties. Furthermore, faculty must provide a supportive learning environment and promote appreciation of mathematics.

"We must equip all of our students–regardless of age, sex, ethnic background, educational goal, occupational goal, personal history or capabilities–to think for themselves and to solve their own problems and those of society to the very best of their individual abilities. That's what the right mathematics does."

Michael Davidson,
Cabrillo College

This document makes no attempt to define "college-level mathematics," nor does it address the issue of whether courses at the introductory level should be credit bearing (to meet graduation requirements).

Summary

Introductory college mathematics must serve well all college students who are not prepared to study at the calculus level or beyond. This document offers a new paradigm for this level of mathematics education. The standards that follow in Chapter 2 are not a "quick fix" for what is wrong. Rather, they provide a strong and flexible framework for the complete rebuilding of introductory college mathematics.

CHAPTER 2:
Standards for Introductory College Mathematics

These standards provide a new vision for introductory college mathematics–a vision whereby students develop intellectually by learning central mathematical concepts in settings that employ a rich variety of instructional strategies.

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Mathematics and its applications should permeate the undergraduate curriculum. Mathematics programs must demonstrate connections both among topics within mathematics and between mathematics and other disciplines. Introductory college mathematics should link students' previous mathematical experiences with the mathematics necessary to be successful in careers, to be productive citizens, and to pursue lifelong learning.

Adult students entering introductory college mathematics programs today bring a rich diversity of experiences. This diversity challenges educators to define clear goals and standards, develop effective instructional strategies, and present mathematics in appropriate contexts. Institutions, departments, and individual faculty must take active roles in addressing the needs of diverse students, in providing a supportive environment, and in improving curricular and instructional strategies. The standards presented in this chapter unite many different mathematical experiences and guide the development of a multidimensional mathematics program.

The standards are based on research evidence and the best judgment of the educators who contributed to this document. They provide goals for introductory college mathematics programs and guidelines for selecting content and instructional strategies for accomplishing these goals. Given the diversity of students and institutions, it is expected that the methods used to implement the standards will vary across higher education and even within institutions.

Framework for Mathematics Standards

The standards presented in this document are consistent with frameworks presented in other mathematics reform initiatives and are intended to affect every aspect of introductory college mathematics. The standards are in three categories

- **Standards for Intellectual Development** address desired modes of student thinking and represent goals for student outcomes.
- **Standards for Content** provide guidelines for the selection of content that will be taught throughout introductory college mathematics.
- **Standards for Pedagogy** recommend the use of instructional strategies that provide for student activity and interaction and for student constructed knowledge.

This framework for mathematics instruction will enable *all* students to widen their views of the nature and value of mathematics and to become more productive citizens.

"Another obstacle to change is the belief held by many mathematicians that the ultimate result of the current movement to revise teaching methods and curricula will be a watered-down mathematics program that is neither effective nor rigorous. They believe that many of the students who leave these courses will not have the mathematical skills necessary for our society and that mathematics majors will not have the experiences necessary for further study in graduate-level mathematics. Research in mathematics education at the collegiate level should produce evidence that students can develop rich concepts from advanced mathematics as they use technology and learn mathematics in alternative settings."

Mary Kim Prichard,
Mathematics Teacher,
December 1993, p. 747.

Standards for Intellectual Development

At the conclusion of their introductory collegiate studies, all students should have developed certain general intellectual mathematical abilities as well as other competencies and knowledge. Introductory college courses in English, psychology, chemistry, or history attempt to broaden an existing educational foundation. In a similar way, an introductory college mathematics program should help students see mathematics as an enriching and empowering discipline.

"Art and music students at all levels have the opportunity to be creative. Mathematics students should have that same opportunity."

Uri Treisman,
University of Texas, Austin

Standard I-1: Problem Solving

Students will engage in substantial mathematical problem solving.

Students will use problem-solving strategies that require persistence, the ability to recognize inappropriate assumptions, and intellectual risk taking rather than simple procedural approaches. These strategies should include posing questions; organizing information; drawing diagrams; analyzing situations through trial and error, graphing, and modeling; and drawing conclusions by translating, illustrating, and verifying results. The students should be able to communicate and interpret their results.

Emphasizing problem solving will make mathematics more meaningful to students. The problems used should be relevant to the needs and interests of the students in the class. Such problems provide a context as well as a purpose for learning new skills, concepts, and theories.

"For most scientists, what's most important in science education is not the imparting of any particular set of facts (although I don't mean to denigrate factual knowledge), but the development of a scientific habit of mind: How would I test that? What's the evidence for it? How does it relate to other facts and principles? The same, I think, holds true in mathematics education. Remembering this formula or that theorem is less important for most people than is the ability to look at a situation quantitatively, to note logical, probabilistic, and spatial relationships, and to muse mathematically."

John Allen Paulos,
Beyond Numeracy, 1991, p.6.

Standard I-2: Modeling

Students will learn mathematics through modeling real-world situations.

Students will participate in the mathematical modeling of situations from the world around them and use the models to make predictions and informed decisions. Swetz (1991) describes the modeling process as "(1) identifying the problem, including the conditions and constraints under which it exists; (2) interpreting the problem mathematically; (3) employing the theories and tools of mathematics to obtain a solution to the problem; (4) testing and interpreting the solution in the context of the problem; and (5) refining the solution techniques to obtain a 'better' answer to the problem under consideration, if necessary" (pp. 358-359). In some cases, faculty may select problem situations and ask students to collaborate on the development of models. In other cases, students may be asked to evaluate previously developed models. Does the model behave as intended in that the equations fit the assumptions of the model? How well does the model agree with the real world it is supposed to represent? Does the model perform well on a data set different from the one for which it was developed? Whether students develop their own models or evaluate models that are given to them, they should look beyond how well a proposed model fits a set of data and attempt to provide mathematical or scientific reasons for why the model is valid.

Standard I-3: Reasoning

Students will expand their mathematical reasoning skills as they develop convincing mathematical arguments.

Students will regularly apply inductive and deductive reasoning techniques to build convincing mathematical arguments. They will develop conjectures on the basis of past experiences and intuition and test these conjectures by using logic and deductive and inductive proof, by framing examples and counterexamples, and by probabilistic and statistical reasoning. They will explore the meaning and role of mathematical identities, support them graphically or numerically, and verify them algebraically or geometrically. Finally, students will judge the validity of mathematical arguments and draw appropriate conclusions.

Standard I-4: Connecting With Other Disciplines

Students will develop the view that mathematics is a growing discipline, interrelated with human culture, and understand its connections to other disciplines.

If students are to gain a sense that mathematics is a growing discipline, course content must include topics developed since the eighteenth century. Topics such as algorithms needed for computer-based solution processes, the use of probability in understanding chance and randomization, and the applications of non-Euclidean geometries lend themselves to a discussion of who developed the ideas, when they were developed, and what kind of human endeavors motivated their development. Students will need to research sources other than standard mathematics textbooks to determine how mathematics provides a language for the sciences; plays a role in art, music, and literature; is applied by economists; is used in business and manufacturing; and has had an impact on history.

If students are to gain a sense that mathematics is a growing discipline, the course content must include topics developed since the eighteenth century.

Standard I-5: Communicating

Students will acquire the ability to read, write, listen to, and speak mathematics.

Students will acquire the skills necessary to communicate mathematical ideas and procedures using appropriate mathematical vocabulary and notation. Students will learn to read and listen to mathematical presentations and arguments with understanding. Furthermore, mathematics faculty will adopt instructional strategies that develop both oral and written communication skills within a context of real applications relevant to the particular group of students. As students learn to speak and write about mathematics, they develop mathematical power and become better prepared to use mathematics beyond the classroom.

"Writing can help mathematics students in many different ways. Students who are required to write must do considerable thinking and organizing of their thoughts before they write, thus crystallizing in their minds the concepts studied. After completion of the written work, it is then available for their own use in later studies, and it may also be shown to other students who have difficulty with the same concept. Finally, the entire process will give students valuable practical experience in expressing their thoughts in writing, a skill that they will most certainly need in any future position of responsibility."

Marvin L. Johnson,
Mathematics Teacher,
February 1983, p. 117.

Standard I-6: Using Technology

Students will use appropriate technology to enhance their mathematical thinking and understanding and to solve mathematical problems and judge the reasonableness of their results.

Students will develop an ability to use technology to enhance their study of mathematics in two ways. First, technology can be used to aid in the understanding of mathematical principles. Shoaf-Grubbs (1994) found that graphing calculators provide "a means of concrete imagery that gives the student new control over her learning environment and over the pace of that learning process. It relieves the need to emphasize symbolic manipulation and computational skills and supports an active exploration process of learning and understanding the concepts behind the mathematics" (p. 191). In general, students can use technology to test conjectures, explore ideas, and verify that theorems are true in specific instances. For example, students can solve quadratic equations and inequalities graphically and then use their knowledge of the graphical solution to clarify the algebraic approach (Hector, 1992).

Second, students will use technology naturally and routinely as a tool to aid in the solution of realistic mathematical problems. "Those who use mathematics in the workplace–accountants, engineers, scientists– rarely use paper-and-pencil

procedures anymore. . . . Electronic spreadsheets, numerical analysis packages, symbolic computer systems, and sophisticated computer graphics have become the power tools of mathematics in industry" (NRC, 1989, p.1). In addition, graphing calculators, dynamic geometry software, matrix software, and statistical packages should be included among the technology staples to be used by students.

Technology should be used to enhance the study of mathematics but should not become the main focus of instruction. The amount of time that students spend learning how to use computers and calculators effectively must be compatible with the expected gain in learning mathematics.

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Standard I-7: Developing Mathematical Power

> **Students will engage in rich experiences that encourage independent, nontrivial exploration in mathematics, develop and reinforce tenacity and confidence in their abilities to use mathematics, and inspire them to pursue the study of mathematics and related disciplines.**

All students will have opportunities to be successful in doing meaningful mathematics that fosters self-confidence and persistence. They will engage in solving problems that do not have unique answers but, rather, provide experiences that develop the ability to conduct independent explorations. At the same time, they will learn to abstract mathematical principles in order to promote transfer of problem-solving strategies among a variety of contexts (Druckman & Bjork, 1994) and to better appreciate mathematics as a discipline. Furthermore, they will develop an awareness of careers in mathematics and related disciplines and have a vision of themselves using mathematics effectively in their chosen fields.

Standards for Content

Mathematics education has traditionally focused on content knowledge. Within this tradition, "knowing mathematics" meant knowing certain pieces of subject matter. This document takes the position that knowing mathematics means being able to do mathematics and that problem solving is the heart of doing mathematics. The successful problem solver can view the world from a mathematical perspective (Schoenfeld, 1992).

"The view of mathematical knowing as a practice (not in the sense of drill-and-practice) is supported by recent trends in the philosophy of mathematics. . . . to understand what mathematics is, one needs to understand the activities or practices of persons who are makers or users of mathematics."

Edward A. Silver in *Selected Lectures from the 7th International Congress on Mathematical Education* (Quebec, August 17-23, 1992), p. 317.

Students gain the power to solve meaningful problems through in-depth study of specific mathematics topics. When presented in the context of applications, abstract topics grow naturally out of the need to describe or represent the patterns that emerge. In general, emphasis on the meaning and use of mathematical ideas must increase, and attention to rote manipulation must decrease.

The content standards that follow are not meant to outline a set of courses. Rather, they are strands to be included in an introductory mathematics program in whatever structural form it may take. The specific themes were selected so that adult students can develop the knowledge and skills needed to function as productive workers and citizens as well as be equipped to pursue more advanced study in mathematics and other disciplines.

Standard C-1: Number Sense

> **Students will perform arithmetic operations, as well as reason and draw conclusions from numerical information.**

Number sense includes the ability to perform arithmetic operations, to estimate reliably, to judge the reasonableness of numerical results, to understand orders of magnitude, and to think proportionally. Suggested topics include pattern recognition, data representation and interpretation, estimation, proportionality, and comparison.

Standard C-2: Symbolism and Algebra

Students will translate problem situations into their symbolic representations and use those representations to solve problems.

Students will move beyond concrete numerical operations to use abstract concepts and symbols to solve problems. Students will represent mathematical situations symbolically and use a combination of appropriate algebraic, graphical, and numerical methods to form conjectures about the problems. Suggested topics include derivation of formulas, translation of realistic problems into mathematical statements, and the solution of equations by appropriate graphical, numerical, and algebraic methods.

Standard C-3: Geometry

Students will develop a spatial and measurement sense.

Geometry is the study of visual patterns. Every physical object has a shape, so every physical object is geometric. Furthermore, mathematical objects can be pictured geometrically. For example, real numbers are pictured on a number line, forces are pictured with vectors, and statistical distributions are pictured with the graphs of curves. Modern dynamic geometry software allows for efficient integration of geometric concepts throughout the curriculum using geometric visualization.

Students will demonstrate their abilities to visualize, compare, and transform objects. Students will develop a spatial sense including the ability to draw one-, two-, and three-dimensional objects and extend the concept to higher dimensions. Their knowledge of geometry will enable them to determine particular dimensions, area, perimeter, and volume involving plane and solid figures. Suggested topics include comparison of geometric objects (including congruence and similarity), graphing, prediction from graphs, measurement, and vectors.

"The context engages them [the students], and when they are engaged they think. . . . We have a lot of sterile problems, like 'add these monomials.' Well, they got all kind of weird answers on that because it doesn't mean anything to them. . . . Mathematics interfaces with ordinary life in so many ways we don't have to be stilted in formulating problems for students."

Gloria Gilmer, quoted in *Multiculturalism in Mathematics: Historical Truth or Political Correctness* by Allyn Jackson. Article appears in *Heeding the Call for Change*, MAA, 1992, pp. 121-132.

Standard C-4: Function

Students will demonstrate understanding of the concept of function by several means (verbally, numerically, graphically, and symbolically) and incorporate it as a central theme into their use of mathematics.

Students will interpret functional relationships between two or more variables, formulate such relationships when presented in data sets, and transform functional information from one representation to another. Suggested topics include generalization about families of functions, use of functions to model realistic problems, and the behavior of functions.

Standard C-5: Discrete Mathematics

Students will use discrete mathematical algorithms and develop combinatorial abilities in order to solve problems of finite character and enumerate sets without direct counting.

"Discrete mathematics is not a competitor of the traditional curriculum, nor is it a revolution that will radically change the way we teach or the textbooks from which we teach. It simply broadens and enriches the mathematics curriculum."

Eric W. Hart, in *Discrete Mathematics Across the K-12 Curriculum: 1991 Yearbook of the National Council of Teachers of Mathematics*, p. 75.

Problem situations in the social and behavioral sciences, business, computing, and other areas frequently do not exhibit the continuous nature so readily treated by techniques traditionally studied in introductory college mathematics. Rather, the problems involve discrete objects and focus on determining a count (Dossey, 1991; Hart, 1991). This standard does not imply that recently developed college courses in discrete mathematics are included in introductory college mathematics. Such courses commonly require precalculus or calculus as prerequisites. The standard echoes the recommendations made in the NCTM *Standards* (NCTM, 1989) and in *Reshaping College Mathematics* (Siegel, 1989); namely, the conceptual framework of discrete mathematics should be integrated throughout the introductory mathematics curriculum in order to improve students' problem-solving skills and prepare them for the study of higher levels of mathematics as well as for their careers. Topics in discrete mathematics include sequences, series, permutations, combinations, recursion, difference equations, linear programming, finite graphs, voting systems, and matrices.

Standard C-6: Probability and Statistics

Students will analyze data and use probability and statistical models to make inferences about real-world situations.

The basic concepts of probability and descriptive and inferential statistics should be integrated throughout the introductory college mathematics curriculum at an intuitive level. Students will gather, organize, display, and summarize data. They will draw conclusions or make predictions from the data and assess the relative chances for certain events happening. Suggested topics include basic sampling techniques, tabulation techniques, creating and interpreting charts and graphs, data transformation, curve fitting, measures of center and dispersion, simulations, probability laws, and sampling distributions.

Standard C-7: Deductive Proof

Students will appreciate the deductive nature of mathematics as an identifying characteristic of the discipline, recognize the roles of definitions, axioms, and theorems, and identify and construct valid deductive arguments.

"Nothing sets mathematics apart from other scholarly pursuits like its reliance on proofs, reason, and logical deduction."

William Dunham, *The Mathematical Universe*, 1994, p. 115.

The dependence of mathematics on deductive proof sets it apart as a unique area of human endeavor. While not being the main focus of instruction in introductory college mathematics, mathematical proofs, including indirect proofs and mathematical induction, will be introduced where they will enhance student understanding of mathematical concepts. Students will engage in exploratory activities that will lead them to form statements of conjecture, test them by seeking counterexamples, and identify and, in some instances, construct arguments verifying or disproving the statements.

STANDARDS FOR PEDAGOGY

One of the most widely accepted ideas within the mathematics community is that students should understand mathematics as opposed to thoughtlessly grinding out answers.

> But achieving this goal has been like searching for the Holy Grail. There is a persistent belief in the merits of the goal but designing school learning environments that successfully promote understanding has been difficult. (Hiebert & Carpenter, 1992, p. 65)

Constructivism [see Crocker (1991)], which has become a popular theory for linking teaching to student learning, is based on the premise that knowledge cannot be "given" to students. Rather, it is something that they must construct for themselves. However, Resnick and Klopfer (1989) are quick to point out that constructivism does not imply that faculty should get out of the way and let students learn by themselves. All of the traditional questions remain: "how to present and sequence information, how to organize practice and feedback, how to motivate students, how to integrate laboratory activities with other forms of learning, and how to assess learning" (p. 4). "The goal is to stimulate and nourish students' own mental elaborations of knowledge and to help them grow in their capacity to monitor and guide their own learning and thinking" (p. 4).

"Because the classroom is such a familiar and uncomplicated place for most people, it's hard to see that developing and executing a good curriculum is about as simple as composing and performing a good symphony."

J. Schorr, in *Phi Delta Kappan*, 1993, p. 317.

While constructivist theories may be interpreted differently by different educators and accepted to varying degrees, Brophy and Good (1986) point out that educational research shows that instructional strategies, be they constructivist or not, have a dramatic impact on what students learn. Two themes cut across research findings: "One is that academic success is influenced by the amount of time that students spend on appropriate academic tasks. The second is that students learn more efficiently when their teachers structure new information for them and help them to relate it to what they already know" (p. 366).

The standards for pedagogy that follow are compatible with the constructivist point of view. They recommend the use of instructional strategies that provide for student activity and student-constructed knowledge. Furthermore, the standards are in agreement with the instructional recommendations contained in *Professional Standards for Teaching Mathematics* (NCTM, 1991).

Standard P-1: Teaching with Technology

> **Mathematics faculty will model the use of appropriate technology in the teaching of mathematics so that students can benefit from the opportunities it presents as a medium of instruction.**

The use of technology is an essential part of an up-to-date curriculum. Faculty will use dynamic computer software to aid students in learning mathematics concepts and will model the appropriate use of technology as tools to solve mathematical problems. The effort spent on teaching students to use technology should be an investment in their future ability to use mathematics. Emphasis should be placed on the use of high-quality, flexible tools that enhance learning and tools they are likely to encounter in future work.

"The graphing calculator is a wonderful tool for exploring and learning. It allows me to 'play around' by comparing graphs and changing variables. It gives an immediate picture of a particular function. It allows a lot of 'what if?' situations–that is great!"

A student's comment

In addition, faculty will use technology as a medium of instruction. Instructional media such as videotapes and computers allow students to progress at their own pace and make mistakes without fearing peer or professional judgment.

The use of technology within the instructional process should not require

more time. In fact, the use of technology, coupled with a decreased emphasis in some traditional content areas, should provide the time that is needed to implement the needed reforms in mathematics education.

Standard P-2: Interactive and Collaborative Learning

Mathematics faculty will foster interactive learning through student writing, reading, speaking, and collaborative activities so that students can learn to work effectively in groups and communicate about mathematics both orally and in writing.

Mathematical literacy is achieved through an understanding of the signs, symbols, and vocabulary of mathematics. This is best accomplished when students have an opportunity to read, write, and discuss mathematical problems and concepts (NCTM, 1989). The following types of experiences will be encouraged in college classrooms: cooperative learning (Crocker, 1992; Becker & Pence, 1994); oral and written reports presented individually or in groups; writing in journals; open-ended projects; and alternative assessment strategies such as essay questions and portfolios (Leitzel, 1991; NCTM, 1991).

Standard P-3: Connecting with Other Experiences

Mathematics faculty will actively involve students in meaningful mathematics problems that build upon their experiences, focus on broad mathematical themes, and build connections within branches of mathematics and between mathematics and other disciplines so that students will view mathematics as a connected whole relevant to their lives.

"An eminent mathematician once remarked that he was never satisfied with his knowledge of mathematical theory until he could explain it to the next man he met."

Philip E. B. Jourdain, *The Nature of Mathematics*. Reprinted in *The World of Mathematics* by James R. Newman, 1956, p. 4.

Mathematics must not be presented as isolated rules and procedures. Students must have the opportunity to observe the interrelatedness of scientific and mathematical investigation and see first-hand how it connects to their lives. Students who understand the role that mathematics has played in their cultures and the contributions of their cultures to mathematics are more likely to persevere in their study of the discipline. Making mathematics relevant and meaningful is the collective responsibility of faculty and producers of instructional materials. Administrators have the responsibility of supporting faculty in this effort.

Standard P-4: Multiple Approaches

Mathematics faculty will model the use of multiple approaches—numerical, graphical, symbolic, and verbal—to help students learn a variety of techniques for solving problems.

Mathematical power includes the ability to solve many types of problems. Solutions to complex problems require a variety of techniques and the ability to work through open-ended problem situations (Pollak, 1987). College mathematics faculty will provide rich opportunities for students to explore complex problems, guide them to solutions through multiple approaches, and encourage both oral and written responses. This will motivate students to go beyond the mastery of basic operations to a real understanding of how to use mathematics, the meaning of the answers, and how to interpret them (NRC, 1989).

Standard P-5: Experiencing Mathematics

Mathematics faculty will provide learning activities, including projects and apprenticeships, that promote independent thinking and require sustained effort and time so that students will have the confidence to access and use needed mathematics and other technical information independently, to form conjectures from an array of specific examples, and to draw conclusions from general principles.

Mathematics faculty will assign open-ended classroom and laboratory projects. In addition, they will help their institutions form partnerships with area business and industry to develop opportunities for students to have realistic career experiences (Reich, 1993). Such activities will enable students to acquire the confidence to access needed technical information, and independently use mathematics in appropriate and sensible ways.

"A large number of rich contexts for mathematics instruction is now available. . . . The main problem is that of implementation, which requires a fundamental change in teaching attitudes before it can be solved."

Hans Freudenthal, *Revisiting Mathematics* Education, 1991, p. 30.

GUIDELINES FOR ACHIEVING THE STANDARDS

Faculty who teach introductory college mathematics must increase the mathematical power of their students. This power increases the students' options in educational and career choices and enables them to function more effectively as citizens of a global community where the opportunities offered by science and technology must be considered in relation to human and environmental needs. In order to achieve these goals, mathematics education at the introductory level requires reform in curriculum and pedagogy.

The idea that mathematics competence is acquired through a curriculum that is carefully structured to include the necessary content at the appropriate time and the use of diverse instructional strategies is an underlying principle of this document. The following tables provide guidance for change in the content and pedagogy of introductory college mathematics. When items are marked for decreased attention, that does not necessarily mean that they are to be eliminated from mathematics education. Rather, it may mean that they should receive less attention than in previous years, or that their in-depth study should be moved to more advanced courses where they may be immediately applied. On the other hand, increased emphasis must be placed on the items listed in the increased attention column in order to achieve the goals set forth in this document.

GUIDELINES FOR CONTENT	
Increased Attention	**Decreased Attention**
pattern recognition, drawing inferences	rote application of formulas
number sense, mental arithmetic, and estimation	arithmetic drill exercises, routine operations with real numbers
connections between mathematics and other disciplines	presentation of mathematics as an abstract entity
integration of topics throughout the curriculum	algebra, trigonometry, analytic geometry, and so forth, as separate courses
discovery of geometrical relationships through the use of models, technology, and manipulatives	establishing geometric relationships solely through formal proofs
visual representation of concepts; for example, probability as area under a curve; timelines for annuities and interest; tables for logic and electrical circuits	rote memorization and use of formulas
integration of the concept of function across topics within and among courses	separate and unconnected units on linear, quadratic, polynomial, radical, exponential, and logarithmic functions
analysis of the general behavior of a variety of functions in order to check the reasonableness of graphs produced by graphing utilities	paper-and-pencil evaluation of functions and hand-drawn graphs based on plotting points
connection of functional behavior (such as where a function increases, decreases, achieves a maximum and/or minimum, or changes concavity) to the situation modeled by the function	emphasis on the manipulation of complicated radical expressions, factoring, rational expressions, logarithms, and exponents
connections among a problem situation, its model as a function in symbolic form, and the graph of that function	"cookbook" problem solving without connections
modeling problems of chance by constructing probability distributions or by actual experiment	theoretical development of probability theorems

GUIDELINES FOR CONTENT (continued)	
Increased Attention	**Decreased Attention**
collection of real data for analysis of both descriptive and inferential statistical techniques	analysis of contrived data
exploratory graphical analysis as part of inferential procedures	"cookbook" approaches to applying statistical computations and tests which fail to focus on the logic behind the processes
use of curve fitting to model real data, including transformation of data when needed	reliance on out-of-context functions that are overly simplistic
discussion of the meaning of nonzero correlation and the independence of correlations from any implications of cause and effect	blind acceptance of r
use of statistical software and graphing calculators	paper-and-pencil calculations and four-function calculators
problems related to the ordinary lives of students; for example, financing items that students can afford and statistics related to sports participated in by females as well as by males	problems unrelated to the daily lives of most students; for example, investments of large sums of money in savings or statistics related to sports only played by males
matrices to organize and analyze information from a wide variety of settings	requiring a system of equations to be solved by three methods
graph theory and algorithms as a means of solving problems	algebraically derived exact answers

GUIDELINES FOR PEDAGOGY	
Increased Use	**Decreased Use**
active involvement of students	passive listening
technology to aid in concept development	paper-and-pencil drill
problem solving and multistep problems	one-step single-answer problems
mathematical reasoning	memoriziation of facts and procedures
conceptual understanding	rote manipulation
realistic problems encountered by adults	contrived exercises
an integrated curriculum with ideas developed in context	isolated topic approach
multiple approaches to problem solving	requiring a particular method for solving a problem
diverse and frequent assessment both in class and outside of class	tests and a final exam as the sole assessment
open-ended problems	problems with only one possible answer
oral and written communication to explain solutions	required only short, numerical answers, or multiple-choice responses
variety of teaching strategies	lecturing

SUMMARY

These standards provide a new vision for introductory college mathematics—a vision whereby students develop intellectually by learning central mathematical concepts in settings that employ a rich variety of instructional strategies. To provide a more concrete illustration of these standards, the Appendix contains a set of problems that brings them to life.

CHAPTER 3:
Interpreting the Standards

Instruction and the classroom environment will be characterized by caring faculty who exhibit high expectations for students regardless of race, gender, socioeconomic class, or disability.

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The standards for introductory college mathematics described in Chapter 2 provide general guidelines for improving the introductory college mathematics experience for all students. This chapter addresses their implementation in programs for underprepared students, called "the Foundation," as well as in programs for students beyond that level. In particular, the standards imply that

- the mathematical foundation for underprepared students must go beyond the teaching of arithmetic and algebra skills,
- technical programs must prepare students for their chosen occupational fields as well as provide them with flexibility in career and educational choices, and
- mathematics courses for students pursuing bachelor's degrees should not only serve as prerequisites for higher-level mathematics courses but should also provide students with meaningful content that has application to their daily lives and their career choices.

Furthermore, it is assumed that instruction and the classroom environment will be characterized by caring faculty who

- acknowledge and appreciate the diverse experiences of their students;
- exhibit high expectations for students regardless of race, gender, socioeconomic status, or disability;
- are mathematicians who can develop sound instructional strategies and content to meet the needs of all students;
- are available to help students outside the classroom; and
- provide a nonthreatening environment that encourages students to ask questions and take risks.

"The reform of mathematics education in the first two years of college speaks of building mathematical power as a basic goal. In every topic and in every course, students should be discovering the usefulness of mathematics as a means to deal with the world around us. The initial courses available at college can be thought of as general education mathematics which provides sound mathematics instruction and the incorporation of knowledge from other disciplines."

Jack Rotman,
Lansing Community College

On a visit to a nearby four-year college campus, a two-year college mathematics teacher ran into a former student.

"Hi. You probably don't remember me. I was a student in your introductory algebra class four years ago."

The teacher replied, "Why, yes, I do remember you. How are you doing these days?"

"I'm doing really well," the student responded. "I'll be graduating this spring with a degree in business administration. When I first came to college, I didn't think I would amount to anything. I was an adult student returning to college, surrounded by all these kids who knew it all. And to top it all off I had to take algebra, a course I never took in high school! Plus, whatever math I learned in school was long forgotten."

"Well, you have obviously done very well for someone who didn't think he belonged on a college campus," noted the teacher.

"Yes," replied the student. "Even though I didn't believe in my ability to do the work, you did. You challenged, encouraged, cajoled, and made me realize that I could learn just as well as others, if not better. I hadn't thought of getting a four-year degree or going into anything that might mean more math, like business administration. But you helped me to see the possibilities. Thank you. I just wanted to let you know that I appreciate what you did for me and to let you see how far I have come." ■

Students should emerge from foundation courses with the ability and confidence to go on to the study of higher levels of mathematics so that they may use mathematics effectively in their multiple roles as students, workers, citizens, and consumers.

This vignette points out that students come to college with a variety of mathematical backgrounds and career goals. In order to serve this diverse student population better, every institution should make available a common core of mathematics. This common core, called "the Foundation," provides a starting point for well-designed programs that meet the individual needs of the students. The goal of the Foundation is to provide a mathematical basis for pursuing various curricular paths. Students should emerge from foundation courses with the ability and confidence to go on to the study of higher levels of mathematics so that they may use mathematics effectively in their multiple roles as students, workers, citizens, and consumers.

The Foundation should have multiple entry points that depend upon the background of the individual student. Students may master the Foundation by successfully completing high school mathematics based on the NCTM *Standards* (NCTM, 1989), by earning a GED based on the emerging standards for adult mathematics education, by taking mathematics courses in college, or by some combination of the three.

The Foundation must be solidly based on the standards described in Chapter 2, including an emphasis on developing reasoning ability, using technology appropriately, and solving substantial realistic problems. It should include topics found in high school basic mathematics, algebra, and geometry courses and in college developmental mathematics courses, although the emphasis will be markedly different. In addition, the Foundation will include other topics vital to establishing a solid core of mathematics knowledge and skills, including mathematical modeling, functions, discrete mathematics, probability, and descriptive statistics.

While college foundation courses serve a vital purpose, precollege students should be encouraged to acquire the mathematical knowledge of the Foundation at the secondary level whenever possible. Colleges are encouraged to join with secondary schools at local, state, and regional levels to develop and promote programs alerting students to the importance of studying more mathematics in high school. Such programs already exist in a number of states and have proved to be very successful (Demana, 1990).

The Goals

The Foundation plays a critical part in the revitalization of introductory college mathematics. Its goals are to

- help students develop mathematical intuition along with a relevant base of mathematical knowledge;
- integrate numeric, symbolic, functional, and spatial concepts;
- provide students with experiences that connect classroom learning and real-world applications;
- efficiently, but thoroughly, prepare students for additional college experiences in mathematics;
- prepare students to work in groups and independently;
- enable students to construct their knowledge of mathematics through meaningful applications and explorations as well as techniques of reasoning, regardless of their level of preparation;
- provide multiple entry points to meet the needs of students who enter college mathematics at different levels of mathematical sophistication; and
- challenge students, but at the same time foster positive student attitudes and build confidence in their abilities to learn and use mathematics.

"I am certain that as women and members of the working class and other cultures participate more and more in the established mathematics, our societal conception of mathematics will change and our ways of perceiving our universe will expand. This will be liberating to us all."

David Henderson in *Fear of Math*, by Claudia Zaslavsky, 1994, p. 25.

Meeting these goals requires continual assessment of students' readiness for new material and a system for supplying the prerequisite building blocks of the Foundation when the need arises.

The Students

Higher education will continue to serve older students with faded mathematical backgrounds, younger students with inadequate secondary school preparation, and students with learning disabilities and other special circumstances. In addition, the gradual implementation of the NCTM *Standards* (NCTM, 1989) in the school mathematics curriculum will add to the diversity of backgrounds of entering students. Furthermore, a disproportionately high number of minorities have entered higher education without the mathematical background to pursue mathematics-intensive coursework (NRC, 1991). The Foundation should play a key role in ensuring equity in and providing access to mathematics-intensive disciplines. The Foundation must expand the educational opportunities and broaden the career options for all students.

Faculty must instill positive attitudes in their students about mathematics.

Frequently, students enrolled in the Foundation have not been successful in past mathematical experiences; and they do not recognize the roles that mathematics will play in their education, in their careers, and in their personal lives. Faculty must provide meaningful content that will establish the importance of mathematics, use instructional strategies that build confidence in the students' abilities to learn and use mathematics, and provide guidance to ensure that their students use appropriate learning techniques.

The Foundation should not replicate the high school curriculum. There is a subtle, but critical, difference between building a college curriculum around students' needs and building it around their deficiencies.

The Content

The Foundation must be designed for the needs and interests of adult students. While it will include many topics taught in high school, the Foundation should not replicate the high school curriculum. There is a subtle, but critical, difference between building a college curriculum around students' needs and building it around their deficiencies. The greater time constraints, the more focused career interests, and the broader experiences of adult learners, as well as the goals and expectations of postsecondary institutions, necessitate different courses specifically designed for college students.

Traditional developmental courses try to cover every possible skill that students might need in subsequent courses. This coverage is likely to be too shallow to equip students for later study or for applying mathematics outside the mathematics classroom. Instead, faculty should include fewer topics but cover them in greater depth, with greater understanding, and with more flexibility. Such an approach will enable students to adapt to new situations.

The Foundation must fully embrace the use of technology and prepare students for a revised mathematics curriculum at the next level. Course content should emphasize how and when to use technology in balance with paper-and-pencil work and manipulatives.

The content for foundation courses is built around the content standards introduced in Chapter 2.

NUMBER SENSE. Number sense involves the intuitive understanding of the properties of numbers and the ability to solve realistic arithmetic problems using appropriate mathematical tools. The latter includes the use of mental arithmetic and calculators, with a significantly reduced emphasis on paper-and-pencil algorithms. Number sense is developed through concrete experiences. It includes knowledge of basic arithmetic facts and equivalent numerical representations and the ability to estimate answers. For instance, a person with number sense will recognize immediately that the sum of one-half and one-third is slightly less than one, or that $\sqrt{10}$ is close to 3. Likewise, numerically literate people recognize that 25%, 0.25, and 1/4 are equivalent. Such intuition is based not on being able to perform an algorithm, but rather on meaningful experiences with numbers.

"Perhaps Pythagoras was a kind of magician to his followers, because he taught them that nature is commanded by numbers. There is a harmony in nature, he said, a unity in her variety, and it has language: numbers are the language of nature."

J. Bronowski, *The Ascent of Man*, 1976, p. 156.

Number sense includes a conceptual understanding of numerical relationships and operations. Students should be able to use numbers to express mathematical relationships that occur in everyday situations. In particular, they should know how to use percent and proportionality relationships. They should also understand concepts on which arithmetic algorithms are based and be comfortable devising their own methods for performing mental arithmetic. Students with a well-developed number sense will have a basis for building an understanding of algebra and the properties of real numbers.

SYMBOLISM AND ALGEBRA. The study of algebra in the Foundation must focus on modeling real phenomena via mathematical relationships. Students

should explore the relationship between abstract variables and concrete applications and develop an intuitive sense of mathematical functions. Within this context, students should develop an understanding of the abstract versions of basic number properties (which assumes they have acquired a reasonably sophisticated level of number sense) and learn how to apply these properties. Students should develop reasonable facility in simplifying the most common and useful types of algebraic expressions, recognizing equivalent expressions and equations, and understanding and applying principles for solving simple equations.

Rote algebraic manipulations and step-by-step algorithms, which have received central attention in traditional algebra courses, are not the main focus of the Foundation curriculum. Topics such as specialized factoring techniques and complicated operations with rational and radical expressions should be eliminated from foundation courses. The inclusion of such topics has been justified on the basis that they would be needed later in calculus. This argument lacks validity in view of the reforms taking place in calculus and the mathematics being used in the workplace. Extensive recommendations for needed reform in algebra instruction are given in *Algebra for the Twenty-First Century* (Burrill, Choate, Phillips, & Westegaard, 1993).

GEOMETRY AND MEASUREMENT. These topics have received scant attention in the typical introductory college mathematics program (see Albers et. al., 1992, p. 85). However, the ability to visualize and mentally manipulate objects is an essential component of the Foundation. The study of measurement in the Foundation will enable students to use both the U.S. Customary System and the International (Metric) System of measurements in problems and in everyday situations. In addition, students should be able to do unit conversions and apply principles of accuracy and precision.

Geometry will include the study of basic properties of angles, polygons, and circles and the concepts of perimeter, area, and volume for basic plane and solid figures. Dynamic geometry software can enliven and deepen this study. Students should use coordinate geometry to make connections between algebra and geometry. Geometry may also be used as a vehicle to acquaint students with the study of logic and to provide an awareness of valid and invalid forms of argument. Although geometry should not be presented as a series of formal proofs, students should be able to construct simple fundamental proofs.

Right triangle trigonometry should be included with the study of geometry in the Foundation. The topic provides a context to connect arithmetic operations, algebraic formulas, and geometric properties. Furthermore, it acts as a basis for the study of analytic trigonometry by mathematics-intensive majors and for the more advanced trigonometry applications studied by many technical majors.

"Geometry is a vehicle that provides so much of the basic core of knowledge that the student of mathematics should possess–basic geometric facts, structure of the system, applications, a study of two- and three-dimensional spaces (including some dimensional analysis), introduction to forms of argument, introduction to forms of proof, the deductive skills that reach far away places (an author writing in a lucid style or a lawyer preparing a strong court case), building the foundation for the coursework of trigonometry, analytic geometry, or calculus. And the list continues in an era that reflects ultra-modern applications of geometry."

Daniel Alexander,
Parkland College

FUNCTIONS. Through the study of functions, students will be able to compute numerical values for, plot, and interpret the graphs of a variety of basic functions. They should be able to create and identify a variety of functions based on patterns in collected data. Students will analyze functions for periodicity, maximum and minimum values, increasing or decreasing behavior, domain and range, and average rate of change. The study of functions will include the use of their multiple representations in order to solve problems. Students should be able to make connections between the parameters of a function and the behavior of the function. Finally, students should be able to use functions to model real-world relationships.

DISCRETE MATHEMATICS. Discrete mathematics can enliven and enrich the Foundation by presenting some traditional topics from a different perspective.

At the most basic level, students can use procedures such as tree diagrams, Venn diagrams, and permutation and combination formulas as aids to solving counting and probability problems. Students can use the ideas of recursion and difference equations to model phenomena from areas of human endeavor much as differential equations are used at a more sophisticated level. In addition, students can learn how to use matrices to store data and to solve problems involving the data (see Problem 10 in the Appendix). At a more advanced level, students can then proceed to use matrices to solve systems of equations.

Discrete procedures offer faculty the opportunity to integrate and connect topics in mathematics. Choppin (1994) describes a hands-on activity, for example, that connects topics in algebra and geometry and involves the students in pattern discovery, series, estimation, and recursion at an elementary level.

PROBABILITY AND STATISTICS. Foundation courses will help students develop an understanding of concepts from probability and statistics. Students will collect, summarize, and display data in such a way that reasonable conclusions may be drawn. In addition, students will be able to determine basic measures of central tendency and dispersion and be able to solve problems involving random events using basic theoretical probability and simulations. Descriptive statistics offers a particularly meaningful context for arithmetic and algebraic problem solving.

DEDUCTIVE PROOF. Formal deductive proof will not be a major emphasis of foundation courses. However, as pointed out in the discussion on geometry, students should be aware of valid and invalid forms of mathematical arguments. Informal deductive proofs can provide this awareness and at the same time give meaning to the content under discussion.

"Restructure remedial courses for success. Incorporate cooperative learning, peer tutoring and computer assisted instruction as supplements to traditional teaching methods. Have experienced, well motivated, and talented teachers work with students in small classes, and promote group study inside and outside of the classroom setting."

Beverly Anderson,
The AMATYC Review,
Spring 1993, p. 7.

The Pedagogy

The pedagogy used in presenting material in the Foundation should mirror the standards in Chapter 2. Of particular importance in the Foundation is teaching with technology. Mathematics faculty who teach foundation material should make effective use of appropriate technology. Technology should be a routine part of instruction. Paper-and-pencil algorithms, however, should be applied to basic computations that are as easily done with paper and pencil as with a calculator. Graphing calculators and computer software should be used when beneficial or advantageous. Their use is especially helpful for geometry, functions, discrete mathematics, probability, and statistics.

The use of cooperative learning strategies is also critical to providing positive learning experiences. Many students at this level have low self-esteem. Faculty must avoid reinforcing student perceptions that the teacher is the sole authority and that the student cannot learn except through the teacher. As faculty take on the role of a coach, rather than that of an authority figure, and as students learn to work together, they will begin to realize the mathematical power they possess.

Faculty teaching the Foundation will have to walk a finer line than those in the other areas. They will have to be compassionate enough to help students work through their frustrations but show enough "tough love" to encourage them to become independent thinkers and help them realize that sustained effort will be required to truly master the material.

Students entering foundation courses bring with them some knowledge of mathematics. Faculty should help students build on this knowledge and recognize its value. Faculty should use manipulatives and other concrete models of

mathematics phenomena to help students make the transition from concrete to abstract thinking.

The Foundation need not be tied to traditional course structures. Goldblatt (1994) suggests that traditional remedial courses be replaced by courses that introduce students "to new areas of mathematics that do not require a highly developed skill in either algebra or arithmetic. Topics such as elementary probability and statistics, game theory, linear programming, and symbolic logic introduce new fields of study from the world of mathematics in ways that do not require the students to retrace the steps of their previous failures" (pp. 7, 9). Skills can be introduced and practiced as they are needed.

Another model for a foundation program centers on students who have at one time gained a necessary level of mathematical sophistication but need to review previously learned skills and techniques. This model calls for placing the students in courses beyond the Foundation level (e.g., precalculus, technical mathematics, and statistics). Then, as the need arises, underprepared students are given special instruction and assignments either by faculty, or as part of a program organized in an academic support center. This approach provides students with the mathematical prerequisites they lack, while involving them in mathematics more immediately relevant to their career goals.

The strategies that are used to implement the Foundation must accommodate students with disabilities and other special needs. For example, flexible scheduling is an important consideration. Students with family or work responsibilities frequently need to attend class in the evenings or on weekends. Students with disabilities or students who have been away from coursework for several years may need an extended time frame in order to complete course requirements. Haney and Testone (1990) describe an after-semester workshop program that allows students to complete course requirements between college semesters.

"One of the aspects of the class that I have particularly enjoyed has been working in groups. When I came to the class and heard this idea discussed, I was very wary. I was older than most of the people in the class and felt that it might be an isolating experience. After the newness of the concept wore off, I found that working with my classmates was very rewarding. We argued, we laughed, and we griped, but we all learned from sharing one another's ideas and methods. I felt very much accepted, and even sought out by the group, which only increased my positive feeling about the experience."

A student's comment

Increased and Decreased Attention

This vision of the Foundation mandates changes in emphasis as well as in content and pedagogy. Traditionally, mathematics at the Foundation level has emphasized the teaching of arithmetic and algebra skills and the solving of "textbook" problems. This document calls for a more balanced approach to skill and concept development. Areas that should receive increased attention include

- the active involvement of students in solving real multi-step mathematics problems;
- the introduction of needed skills in the context of real applications;
- mental arithmetic, estimation, geometric properties, and the translation of problem situations into algebraic models;
- the integration of mathematical topics so that students may use a wide range of mathematical content and techniques to solve problems;
- the conceptual understanding of mathematical ideas and the ability to use valid arguments; and
- the appropriate use of technology throughout the curriculum for computational work, graphing, geometry, probability, and statistics.

Areas that should receive decreased attention include

- paper-and-pencil drill with arithmetic algorithms; longhand simplification of polynomial, rational, exponential, and radical expressions; and factoring;
- solving contrived word problems, equations, and inequalities;
- the isolated topic approach to teaching and learning; and
- one-step, single-answer, and single-method template problems.

"Technology can be used to solve mathematical problems provided the user understands the mathematics of the problem."

Task Force

Summary

The Foundation described in this document is radically different from the traditional secondary and developmental curricula. Students who successfully complete a study of this Foundation will have acquired a basic knowledge of mathematics that will give them the ability and confidence to go on to higher levels of mathematics that are needed in their particular areas of study and to become effective citizens in a modern society. This goal can be reached if the faculty of foundational courses engage their students in activities designed to enhance their intuitive understanding of mathematics and their belief in their own ability to do mathematics. Changing the existing curriculum to conform to the guidelines outlined here will be a formidable task. Such a change is needed, however, so that the educational experiences of underprepared students will be more relevant and valuable.

Lynn walked into her office on Monday morning to find a memo marked "URGENT." As she set her briefcase down, she glanced at the contents and was relieved to see that it was the request she had been anticipating. The company that employs Lynn had begun to make a new line of components to be shipped to other companies for use in manufacturing. It is Lynn's job, as a time study analyst, to determine the standard for the amount of time required to package these components.

The memo requested that Lynn provide as soon as possible a standard for placing hardware components in a bag. The components vary in size and weight, and different size bags are used to package the individual orders. It went on to say that the company was already receiving calls for the items and needed to establish standards to price them properly. The standards would be difficult to establish since the three factors all affect the time required to package each order. Lynn was prepared; she had already been on the production line and taken data (a sample of the data appears in Table 1) from which she could prepare the standard. Now she had to analyze the data to develop a formula or system of curves to serve as a model for predicting the time required to bag the components. She would then have to test the model and, if it proved to be satisfactory, write a report. ■

Study No.	Time (min.)	Weight of Components (lb)	Bag Size	No. of Components
1	0.264	6.62	4	11
2	0.130	1.15	2	6
3	0.186	5.61	3	8
4	0.169	2.91	2	6

Table 1. Time Required to Place Hardware in a Bag.

"Workers are less and less expected to carry out mindless, repetitive chores. Instead they are engaged actively in team problem-solving, talking with their co-workers and seeking mutually acceptable solutions."

Mathematics Education, Wellspring of U.S. Industrial Strength, 1989, p.3.

Students in technical programs should be prepared to function effectively in an ever-changing workplace. As the vignette shows, the mathematical tools workers use have changed drastically in the last few years. As a result of implementing computing and networking technology, companies have restructured, pushing decision making down the organizational structure. Technicians often work with their peers in teams and are required to acquire and process information and make decisions based on data formerly available only to engineers and managers. In this data-driven, technologically advanced environment, technicians must be proficient in the application of technology, in mathematics at the level of the Foundation and beyond, and in critical thinking.

The mathematical preparation of technical students should focus on applications. The effectiveness of their education will be very limited, however, if they do not become proficient in performing basic mathematical skills and have

an intuitive understanding of fundamental mathematical principles. The mathematics studied by students as part of their technical programs must support them if their careers change, or if they decide to study additional more sophisticated mathematics.

Collins, Gentry, and Crawley (1993) recommend that two- and four-year colleges and universities should "enhance upward academic mobility (e.g., from Associate of Applied Science to Bachelor of Engineering Technology programs) through articulated curricula with emphasis on applied content and skills. Bridging or transitional programs should be available where necessary" (p. 7). Although the focus of a technical student's mathematical interest may be applications to his or her chosen field, these students should also learn to appreciate the usefulness of mathematics for solving problems from a variety of fields. The development of the ability to reason and to analyze problems from other disciplines will assist technical students in becoming better workers and better informed citizens.

"Students' dispositions for change between the ages of 15 and 20 virtually guarantee that no inflexible system of early tracking can be personally or educationally sound."

Susan Forman and Lynn Arthur Steen in *Focus*, June 1994, pp. 1 and 9.

Technicians need to be flexible workers who can adapt to the changing needs of their jobs. These changes often necessitate the acquisition of additional skills, especially in the application of mathematics. Therefore, the educational environment should encourage students to develop strategies for learning additional mathematics independently in order to change from one job to another and to meet their evolving educational and career objectives.

To accomplish these goals, the content and structure of the mathematics curriculum for technical students must be both rigorous and relevant. Druckman and Bjork (1994), in a summary of research on transfer of learning, point out:

> Although concrete experience is very important, the teaching of abstract principles plays a role in acquiring skills over a broad domain of tasks. . . . A training program that provides learners with varied contexts and general procedures allows them to adapt to new situations not encountered during training. (p. 11)

Mathematics courses must be designed around the concepts and applications that connect topics and make mathematics meaningful. Courses organized around the manipulation of algebraic symbols and routine exercises do little to promote transfer of learning. To prepare students for the world of work, the mathematics they study must be broad based. It must provide the necessary skills and conceptual understanding that will allow for the study of more advanced concepts, as well as the appropriate problem-solving strategies for solving real problems in a variety of contexts and interpreting their results.

The Content

"A strong foundation in mathematics is an absolute requirement for technical education."

Gaining the Competitive Edge, NSF, 1993, p. 19.

"The key issue for mathematics education is not whether to teach fundamentals but which fundamentals to teach and how to teach them," writes Lynn Steen (1990, p. 2). If mathematics courses for technical students are to include realistic problem solving, extended projects, collaborative work, and portfolios, faculty must reexamine the structure and content of the curriculum. In conjunction with professionals from other disciplines and representatives from business and industry, they must decide what mathematics is most important for technical students to learn.

The curriculum should provide substantive mathematical challenges, building upon the Foundation to include an understanding of numeric, algebraic, and geometric topics. Some programs might also include trigonometry, statistics, or an introduction to calculus (Collins et al., 1993). Blending applications and skills, the curriculum should develop intuitive understanding through a combination of

paper-and-pencil and technology-oriented activities. For example, once students understand simple exponential functions and can evaluate them with paper and pencil, a computer or graphing calculator may be used to evaluate and graph more complicated examples. Time can then be spent giving meaning to the solutions of problems involving exponential functions and determining the effect changes in the parameters have on the solutions. Enough attention to proof and formal derivations should be given to provide students with an appreciation of the supporting mathematical theory; however, the focus of the technical curriculum should be on applications.

Central to the mathematics education of technical students is the development of the ability to design and use algorithmic procedures for solving problems. Traditional programs have emphasized continuous mathematics. Technical students should also be introduced to discrete algorithms in such areas as counting and graph theory. Technical applications often involve determining the optimal way to perform certain procedures, and discrete algorithms can be used to determine the solution. Gardiner (1991) warns, however, that using algorithms should not degenerate "into a succession of meaningless routines" (p. 12). At the introductory level, students should be introduced to a small number of central techniques that can be applied to realistic problems in a meaningful manner.

The content of courses designed to provide the mathematics needed by technical students in such areas as health-related services, business, and engineering technologies will vary. All courses should build on the Foundation to meet the needs of students in individual programs. Course proliferation due to excessive customization, however, should be minimized. Programs with similar mathematical needs should enroll students in common mathematics courses in which applications and student projects may be program-specific.

The curriculum should also include applications from the natural sciences. For example, experimenting with various weights on a spring can motivate the study of linear functions, and examining the change in the temperature of a liquid as it cools can motivate the study of exponential functions. Mathematics courses should not only be designed to meet the immediate needs of technical students. Rather, they should also be broad-based and rich in content in order to meet the students' employment and personal needs now and in the future.

Some of the factors that impact the mathematics curriculum for technical students are accrediting agency guidelines, programs at other colleges, recommendations of professional organizations, the views of the mathematics faculty, and experts from other fields. Two-year colleges should also attempt to secure articulation agreements with four-year colleges and universities for students who wish to transfer after obtaining a two-year technical degree. Within these constraints, colleges should design the mathematics curriculum so that technical students will be able to change from one technical program to another, or from a two-year associate in an applied science degree program to a bachelor's degree program, with a minimum of backtracking.

Mathematical content should be introduced in the context of real problem-solving situations. Specific courses should integrate mathematical themes, with less regard for traditional classifications such as algebra, trigonometry, and geometry. In addition, technology should prompt faculty to rethink the presentation of certain topics. For example, the idea of linear function is one topic that deserves emphasis. Rather than teach that topic as an isolated segment in a chapter on functions, it might be motivated by an application in which students have to find a line that best fits some real data. With the aid of technology the discussion could be continued with different data sets, and the models generated could then be used for making predictions. The application would give real meaning to x- and y-intercepts and provide a reason for finding them. Slope would acquire meaning

"The distinction between learning and performance is critical because most training and task contexts differ in some way. . . . Indeed, training cannot generally anticipate the full range of circumstances that will be encountered in task performance, and even anticipated circumstances may be impossible to fully simulate in training. Ideally, a training program should produce the ability to accommodate some degree of variability–both within the task environment and between the training and task environments–as well as establish basic skills required for the task itself."

Daniel Druckman and Robert A. Bjork,
Learning, Remembering, Believing: Enhancing Human Performance,
1994, pp. 25-26.

"Educators, . . . need to work closely with employers and workers to develop an understanding of what kinds of skills employers require in their workers and what kinds of jobs are available in the local labor market."

Robert B. Reich,
Educational Record,
Fall 1993, p. 23.

with particular emphasis given to the units describing the change in one variable in terms of another. The problems could be extended by evaluating residuals. Changing the parameters of the model would then demonstrate the effect of the parameters on the residuals.

Curricular content should undergo continuous evaluation and updating. In industry, technology has led to new expectations of employees. In education, it has led to drastic changes in what can be taught and how. In addition to adjusting to meet the demands of technology, industries are being restructured due to economic trends, government regulations, and political pressure. All these developments will dictate the direction of change in curricular content in courses for technical students.

The Pedagogy

In mathematics courses for technical students, instructional strategies should include

- interactive learning through writing, reading, and collaborative activities;
- projects and apprenticeship opportunities that encourage independent thinking and require sustained effort;
- use of multiple approaches (numerical, graphical, symbolic, and verbal) to solve meaningful problems; and
- the use of interactive and multimedia technology.

"Workers [who are skilled, adaptable, and able to learn quickly] are finding that new, advanced technology is an invaluable asset, the means by which they can add more value to their products and command a bigger paycheck. . . . those [workers] who succeed will not only have to be highly skilled, but also so broadly skilled that they can change jobs and occupations without missing a beat."

Doug Ross,
Community College Journal,
April/May 1994, p. 19.

While the instructional methodology associated with technical programs will not differ significantly from the pedagogy for more general courses, faculty should adapt instructional strategies for particular technical programs where such adaptation will enhance the learning environment. For example, a faculty member teaching mathematics to electronics students can design a laboratory experience that explores sine waves of voltage using an oscilloscope. Mathematics classroom experiences with equipment specific to a technology area may be team-taught with a technology faculty member, or prepared in consultation with a practitioner.

As indicated in *What Work Requires of Schools: A SCANS Report for America 2000* (Secretary's Commission on Achieving Necessary Skills [SCANS], 1991), mathematics should be taught in context. That is, students should learn content while solving realistic problems. A contextual approach will make liberal use of technology and focus as much on what solutions mean as how they were obtained. Using interactive and multimedia tools, the classroom can become an open gateway to the workplace. Classrooms will no longer be bounded by walls, local resources, or a single faculty member's knowledge. Students will be challenged to develop solutions to real problems in a virtual workplace.

Individuals working in business or industry who are qualified teachers can be brought in as adjunct faculty. Such individuals can enhance mathematical instruction for technical students by bringing to the classroom valuable expertise as practitioners in their fields. Care must be taken, however, in the use of part-time faculty, as outlined in Chapter 4. In addition, all faculty should regularly consult with practitioners in the field in order to remain current in the applications of mathematics.

Increased and Decreased Attention

Mathematics courses for technical students should develop mathematical intuition through an understanding of the content and how it may be applied to solving problems. According to the 1993 SCANS report (SCANS, 1993), programs for technical students should place increased attention on

- organizing and processing information,
- estimating,
- working in groups,
- reading technical charts and graphs,
- reading and learning from other technical materials,
- working with formulas computationally and algebraically,
- solving problems from real applications,
- making regular use of appropriate and field-specific technology, and
- communicating results.

Other topics that deserve increased attention in technical programs include statistics; probability; rate change; conversions; difference equations; matrix methods; evaluating the results of numerical computations and graphical displays obtained from computers and graphing calculators; data collection, manipulation, transfer, and analysis; exponential and logarithmic functions; and discrete algorithmic problem-solving strategies. All programs need to ensure that graduates can understand and apply basic principles of statistics and probability.

Topics deserving less attention include determinants and Cramer's rule; trigonometric identities; complicated factoring; graphing functions with paper-and-pencil; Descartes' rule of signs; formulas for finding roots; secant, cosecant, and cotangent functions; radical equations with more than one radical; complex rational expressions; and complex expressions involving exponents.

Summary

Beyond the Foundation, the mathematical needs of technical students may vary according to their field of study. The courses that provide students with the required mathematics, whether they are taught in an integrated approach or not, must provide a broad base of mathematical knowledge. They must also contain the appropriate rigor and depth to allow students to study additional mathematics that their careers may require and to ease the switch from one technical area to another or the transfer from an associate's degree to a bachelor's degree program.

All students need to have experience using mathematics, combined with technology, to solve real-world problems. "Plug-and-chug" drill should not be translated to a computer screen; students must be given substantive exercises that develop mathematical understanding as well as facility with technology and solid understanding of its power and limitations. In addition to being able to arrive at results, technical students must be able to interpret and use them. These goals require faculty to think deeply about what they teach technical students and what teaching strategies are most effective.

"To me, any tracking that keeps young people out of the mathematics pipeline should be discouraged throughout the students' formative years. But after the student makes a decision about his or her career and after the student has completed a common core of mathematics, I see no problem with classes geared toward the aspirations of the students. The statement is made, of course, provided that students have entry to any other tracks at the appropriate level if they change their minds. Tracking at this level comes from the student, and not from anyone who may have low expectations for the student."

Jack price,
NCTM News Bulletin,
December 1994, p. 3.

With major support from a National Science Foundation grant, the University of Illinois hosted a conference for 75 leading mathematics educators on Preparing for the New Calculus in April 1993 (see Solow, 1994, for the conference proceedings). The conference was organized under the assumption that calculus reform is now a reality and high schools and colleges must prepare students in new ways to meet the needs of the new calculus. ■

"At all degree levels [in the mathematical sciences], several patterns are apparent. The numbers of degrees awarded annually have decreased significantly since peaking in the early 1970s but have shown some increases recently. The attrition rates in degree programs are high. Relatively few women, blacks, and Hispanics receive degrees, especially graduate degrees, and non-U.S. citizens are close to achieving a majority among graduate students. There are increased demands for mathematical sciences degrees in the workplace, and these increases are projected to continue."

A Challenge of Numbers, NRC, 1990, p. 52.

Mathematics-intensive majors include mathematics, science, engineering, computer science, economics, and business. Included with the mathematics majors are those students who are preparing to be secondary school or college mathematics teachers. Because students in these programs are required to study calculus, they need mathematics preparation, often termed "precalculus," that goes beyond the Foundation. This section describes the special needs of such students and the role that introductory college mathematics courses can play in helping them to attain their educational goals.

The mathematicians, scientists, engineers, and economists of the future emerge from our mathematics-intensive majors. Faculty must demand quality performance of these students at this level of their education. Excellence does not materialize suddenly in calculus or upper-division mathematics, or when these students begin designing bridges, telescopes, or business strategies; it emerges from sustained work in challenging courses that offer a rich variety of mathematical experiences.

The job market for mathematics-intensive majors has become increasingly competitive. Two major factors in career success are flexibility of outlook and approach and the ability to work in teams. Introductory college mathematics programs must provide multiple problem-solving methods, promote teamwork, and emphasize meaningful problems that require extensive thought and develop insight.

The Content

The precalculus curriculum must prepare students to be successful in a wide variety of calculus programs. The topics outlined below are basic to the modeling and problem-solving standards that should form the heart of precalculus education. While not departing from concerns about mathematical processes and techniques, more emphasis should be placed on developing student understanding of concepts, helping them make connections among concepts, and building their reasoning skills in preparation for higher-level courses in mathematics and related fields.

FUNCTIONS. "Having a sense for number and having a sense for functions are among the most important facets of mathematical thinking" (Eisenberg & Dreyfus, 1994, p. 45). Just as a sense for number allows students to reason efficiently with numerical information, a sense for functions allows students to

gain insights into the relationships among variables in problem-solving situations. While students begin their study of functions before reaching the precalculus level of mathematics, Demana (1994) points out that building a strong understanding of functions is a common theme in college precalculus reform projects. At this level, students learn to treat functions as objects and to reason formally about operations on sets of functions (Thompson, 1994).

Mathematics-intensive programs should include the study of linear, power, polynomial, rational, algebraic, exponential, logarithmic, trigonometric, and inverse trigonometric functions. Students should also develop a general understanding of the relationship of a function to its inverse, if that inverse exists. Although the rectangular form for functions should be emphasized, parametric and polar representations should also be studied.

Students should be able to categorize and organize functions into families and explore their properties. In addition, students should be able to use the algebra of functions and analyze functions graphically and numerically: find zeros, locate intervals where the function is increasing or decreasing, describe the concavity of the function, and approximate extreme values.

DISCRETE MATHEMATICS. Recognition and use of patterns, including those dealing with aspects of the very large and the very small, are essential to problem solving in mathematics. Numerical techniques that have always played a role in estimation and measurement take on new significance with the increasing use of technology. Spreadsheets and graphing packages allow students to use iterative processes to approximate solutions and guide investigations. The ability to build templates and then change the values of the parameters enables students to investigate the effects of these changes on the model. Recursion is also an important technique in building models for many applications. At the precalculus level, students can use difference equations to model phenomena that are studied with differential equations in calculus, including population growth, Newton's laws of heating and cooling, and simple harmonic motion.

Matrices are very powerful mathematical tools that are often overlooked in introductory college mathematics. Students can use matrices to store data, represent graphs, represent geometric transformations, or solve systems of linear equations. Examples of the use of matrices in modeling situations include representations of production strategies and probability distributions.

STATISTICS. The standards emphasize using real data and probabilistic concepts. Data analysis is especially important for students in mathematics-intensive programs. Students should work with real data, transform them to linearize them, undo the transformation to produce a function that models the original data, and make inferences based on the results. Students should gain experience with probabilistic models, including normal and binomial distribution models, and use Monte Carlo simulations to provide information on processes that cannot be assessed deterministically. These topics cannot all be integrated into a single calculus preparation course. In particular, statistical inference requires separate attention. Students who have not studied introductory statistics prior to studying calculus should do so before they graduate.

"The calculus reform movement is demonstrating that we should come to expect greater fragmentation in the mathematics curriculum. It is possible for different schools to offer quite different calculus courses that are successful locally and transferable (in several senses) globally–transferable to other institutions for credit, and transferable to other disciplines in terms of students being able to apply the mathematics they have learned. We will likely see this trend extending to other parts of the curriculum in the future."

Sheldon Gordon,
Suffolk Community College

The Pedagogy

The standards for pedagogy in Chapter 2 provide appropriate instructional guidelines for mathematics-intensive majors. In particular, these standards

advocate building connections with other fields and approaching problem solving with a variety of strategies. Such pedagogical techniques prepare students to use mathematics effectively in their own fields of study. For mathematics majors, these kinds of experiences foster a broad outlook on the field and provide opportunities for developing deeper insight. For prospective secondary school mathematics teachers, the pedagogical strategies model those that the students will use when they become classroom teachers.

The pedagogy standards also place emphasis on cooperative learning and the use of technology to encourage student investigation, discovery, and insight. The use of technology is especially critical for students in mathematics-intensive programs, who will enter a workplace where expertise in technology will be assumed. Students must become sufficiently comfortable with graphing calculators and computers so that they automatically reach for them when, and only when, these mathematical tools offer a better or quicker way to solve a problem. To attain this technological confidence, students must use these tools in class, at home, and on examinations other than those specifically designed to test for knowledge of basic skills. Students will never have a job in which they will be restricted from using these "tools of the trade." Faculty who restrict calculator or computer use on examinations should reexamine their policies and ask what purpose the restrictions serve. If calculator use on an examination yields an automatic "A" grade, faculty are asking the wrong questions, or teaching the wrong material, or both. For example, instead of asking students to graph a polynomial function, a task easily done with a graphing calculator, faculty could provide a graph or a set of data points and ask students to write an equation that would generate that type of graph. This approach requires deeper understanding of graphing and also assesses understanding of modeling.

"Calculus reform has come to focus more on how calculus is taught and less on what is actually taught. . . . The overall focus on raising students' conceptual understanding, problem solving skills, analytic ability, and transferability of calculus skills to work in other disciplines has led to general changes not in the list of topics and techniques covered, but in how these topics are developed."

Alan C. Tucker and James R. C. Leitzel, *Assessing Calculus Reform Efforts*, 1994, p. 26.

Increased and Decreased Attention

Increased attention should be given to providing students with a global view of the concept of a function. Students should be able to

- distinguish between classes of functions;
- understand periodic behavior and properties that cut across classes of functions, such as transformations;
- use functions in modeling situations;
- use exponential and logarithmic functions in problem solving in a variety of applications;
- use decomposition of functions to analyze the behavior of complicated functions; and
- interpret the behavior of graphs of functions near asymptotes and for very large and very small values of the variable.

Increased attention should be given to determining the real roots of any

equation by a combination of graphical and numerical methods. Similarly, graphing calculator or computer features should be used to solve systems of equations.

Faculty should introduce these concepts and techniques in the context of solving real problems. Such problems should lend themselves to solutions by a variety of strategies and should include student-generated data and data from outside sources. The properties of plane and solid figures offer a rich source of meaningful applications.

Decreased attention should be given to such traditional topics as

- graphing functions with paper and pencil;
- the cotangent, secant, and cosecant functions;
- reduction formulas and the proofs of complicated trigonometric identities;
- conic sections (especially complex algebraic manipulations);
- linear interpolation and other table manipulations;
- partial fractions and factoring beyond the level of the Foundation;
- equation-solving strategies such as the upper- and lower-bounds theorem and Descartes' rule of signs; and
- drill and practice on routine exercises and contrived applications.

Brief descriptions of several precalculus reform projects are given in *Preparing for a New Calculus* (Solow, 1994).

"Traditionally, college students taking precalculus have previously been exposed to basic ideas from an algebraic point of view. The students seem to become overconfident and/or bored and do not work consistently. The graphical approach adds new light to known concepts, thereby allowing the students to gain depth while maintaining their interest. The hands-on approach and the ability to check their answers that the graphing calculator provides, seem to have further enhanced students' motivation."

Antonio R. Quesada and Mary E. Maxwell,
Educational Studies in Mathematics,
Vol. 27, 1994 p. 213.

Summary

Faculty who teach students pursuing mathematics-intensive majors face several challenges. They must adapt the curriculum to prepare students with diverse mathematical backgrounds and learning styles for reformed calculus courses. They need to vary their teaching strategies in response to research on student learning. Even classroom layouts must change. Classrooms must be designed to accommodate cooperative learning, laboratory investigations, and computers. Faculty must face these challenges with renewed enthusiasm to increase the effectiveness of precalculus mathematics and subsequent calculus programs. The entire mathematics community must cooperate to transform the mathematics "filter" into a "pump."

LIBERAL ARTS PROGRAMS

Curricula in the humanities and social sciences are grouped under the term "liberal arts programs." Typically, these programs have been planned with little emphasis on mathematics. Yet as mathematics and its applications become increasingly pervasive in society, the need for all citizens to understand and use mathematical ideas increases as well. All college educated individuals should be competent and confident in the Foundation presented in this chapter. As indicated in the following description from *National Goals for Education* (U.S. Department of Education, 1990), however, additional mathematics will be needed by all college graduates:

> Our people must be as knowledgeable, as well trained, as competent, and as inventive as those in any other nation. All our people, not just a few, must be able to think for a living, adapt to changing environments, and understand the world around them. (p. 1)

"To appreciate mathematics fully, one must recognize it as a vital, on-going discipline, one that is practiced by a worldwide community of dedicated, sometimes passionate, and frequently brilliant scholars. It is a surprise to many that mathematics is a living, changing, developing subject. A true appreciation of mathematics requires some knowledge of contemporary developments."

Reshaping College Mathematics, edited by Lynn A. Steen, 1989, p. 110.

The following vignette illustrates the kind of mathematics liberal arts programs can use to prepare students to "think for a living" and "adapt to changing environments."

A mathematics faculty member invited the Director of Supply Management of a large television manufacturer to speak to his class. The director described his own educational background in mathematics as being below calculus and stated that he had to use a considerable amount of mathematics to solve a variety of work-related problems. He then presented a spreadsheet of industry data on the yearly sales of televisions and asked the class to predict what would happen to the sales of television sets the next year; this problem was important to his position and the company. He described briefly some of the company decisions influenced by the prediction. Upturns in sales might indicate the need for a new factory, while downturns would result in a backlog of finished sets which would have to be stored. Following questions, the class divided into groups with the assignment to analyze the data, make observations and hypotheses, propose a prediction model, and write group reports. Motivation and interest were high.

The group made oral reports to the director in a later class period. They observed that downturns in sales occurred at about the same time as the oil embargo and the Gulf War. They built several linear prediction models after making scatter diagrams. Some tried polynomial and exponential curve fitting. All used statistical software in a variety of ways. The students and faculty member were very pleased with the director's feedback on the oral presentations. ■

The students in the vignette used substantial mathematics including the graphical presentation of data as well as linear and nonlinear regression. Students were able to see that mathematical results need to be interpreted in contexts that involve history, economics, and other disciplines. It was also helpful for students to tackle a problem posed by someone who did not have extensive background in mathematics. Such an experience enables students to develop a broader view of the nature of mathematics and the role it plays in the world.

The Content

The traditional college algebra or precalculus courses, which are primarily

designed to prepare students for calculus, do not provide the breadth of mathematics needed by liberal arts students (Sons, 1995). Haver and Turbeville (1995) describe the goals of a capstone course that they developed for nonscience majors as follows:

> The goals of the course are to develop, as fully as possible, the mathematical and quantitative capabilities of the students; to enable them to understand a variety of applications of mathematics; to prepare them to think logically in subsequent courses and situations in which mathematics occurs; and to increase their confidence in their ability to reason mathematically. (p. 46)

"Mathematics departments need to begin a dialogue with the nonscience departments concerning the needs of their majors and what they can reasonably expect mathematics courses to teach their majors."

James Woeppel, *UME Trends*, March 1993, pp. 4-5.

The Haver and Turbeville proposal fits the guidelines for a "foundation" course recommended by the CUPM Subcommittee on Quantitative Literacy (Sons, 1995). Note that the term "foundation" in the CUPM report is being used to denote a portion of a broad quantitative literacy program aimed at developing "capabilities in thought, analysis, and perspective" (p. 12). The mathematical content of that "foundation" is intended to be beyond what is normally studied in high school.

It is not appropriate or possible to be prescriptive about the specific mathematical content needed in a liberal arts program. For example, a history major is not necessarily preparing to be a historian, nor is every psychology major preparing to be a psychologist. Some students will seek jobs immediately after earning associate's or bachelor's degrees, while others will enter graduate school. Nevertheless, liberal arts students are likely to encounter formulas, graphs, tables, and schematics and to be asked to draw conclusions from them in the course of work or study.

On a personal level, these students may need to decide whether it is safe to swim in or eat fish from a local waterway or to evaluate different financing options when making major purchases involving amortization. On the job, liberal arts graduates may be asked to predict whether the need for services will increase and necessitate the hiring of additional personnel; or, as in the vignette, they may be asked to translate raw mathematical information into symbolic, visual, or verbal forms. Students going into research-oriented fields or on to graduate school will need background on how to interpret and conduct research. And virtually all workers must deal with computers and calculators at some level. Mathematics courses provide the natural and appropriate place for such learning to occur. As citizens, workers, and students, liberal arts majors will be called upon to use technology to solve problems that require mathematical methods.

Each institution has the responsibility of evaluating local needs and resources to determine how best to educate liberal arts majors in mathematics beyond the Foundation. This additional study might take the form of one or two courses or a series of modules in an interdisciplinary sequence. Small institutions may have to design a curriculum to serve all majors in the same courses.

Faculty must question how well the mathematics requirements in liberal arts programs at their institutions prepare students to function in today's world.

Faculty must question how well the mathematics requirements in liberal arts programs at their institutions prepare students to function in today's world. The ultimate goal for such courses is "to instill in the student an appreciation of mathematics. For this to occur, students must come to understand the historical and contemporary role of mathematics and to place the discipline properly in the context of other human intellectual achievement" (Goldstein, 1989, p. 110). This

goal should not be interpreted to mean that courses for liberal arts students must be broad surveys of mathematical topics, nor that historical topics should dominate. Rather, the emphasis should be on the importance of doing mathematics in order to learn mathematics.

Survey courses have been popular for liberal arts students. Although liberal arts students need a wide variety of content, the curriculum should enable them to see mathematics in a few specific contexts studied in depth. Such topics should highlight the usefulness of mathematics and offer problem-solving opportunities in a variety of liberal arts disciplines. Examples of "big ideas" to be studied in depth include randomness as it relates to uncertainty, probability, and sample selection and social choice and decision making as they relate to voting systems, fair division, and game theory. The topics selected should present fresh mathematics to students rather than a rehash of previously studied topics. Furthermore, the topics should be interesting and challenging for students with a variety of mathematical backgrounds.

Liberal arts students frequently complete their study of mathematics by taking an introductory statistics course. Such a course should go beyond the level of statistical topics covered in the Foundation and include modern techniques for data analysis, estimation, hypothesis testing, and regression. Graduates of such a course should be able to address effectively the problem posed in the introductory vignette.

"Technology can save us or sink us in the classroom. Creative applications of technology can restore much of the thrill of exploration by giving even our less skillful students tools to take them where they could not have easily gone before. But we must learn how to pass on the 'mathematical mind' to our students without the drill and manipulations. We must reinfect them with the excitement of discovery, with the dramatic power of analytical reasoning. We have a lot to learn, but we stand at the door to a new era in mathematics education."

Michael Davidson,
Cabrillo College

The Pedagogy

Every college graduate should be able "to analyze, discuss, and use quantitative information; to develop a reasonable level of facility in mathematical problem solving; to understand connections between mathematics and other disciplines; and to use these skills as an adequate base of life-long learning" (MAA, 1993, p. 8). Keith and Leitzel (1994) suggest "more student interaction, problem-solving and understandable applications" (p. 6) as a productive approach for producing quantitatively literate graduates. All of these recommendations support the standards for pedagogy presented in Chapter 2.

For students to become active users of mathematics, the role of the teacher must change from a sage who hands down knowledge to a coach who provides guidance and support. Faculty need to build self-confidence in students. Some liberal arts majors may not see themselves as doers of mathematics; indeed, they may be fearful of mathematics. Initial class activities must be designed to lead to student success. Cooperative learning experiences should be devised to use the differing strengths of students. Frequent praise for finding alternative solutions to problems will break down the belief that there is one right answer and one right way of solving a problem. Students should be expected to use technology to solve problems and to write project reports, just as they would do at work. Students who are shy about speaking can use short journal entries in order to express their ideas. Then faculty can have students read their journal entries, stressing that sharing ideas helps both the one who developed the idea and those who listen to it.

The intellectual development standard of modeling has special applicability for liberal arts majors. Mesterson-Gibbons (1989) points out that mathematical modeling can be approached pedagogically along a continuum. At one end of

the continuum faculty assign open-ended case studies in which students select a problem, do research, collect data, and pursue a variety of paths to create a mathematical model. At the other end of the continuum, students are presented with a problem that illustrates an already developed model. The students may be asked for interpretations or predictions based on the model.

The problem posed in the introductory vignette is near the middle of this continuum. It was not possible to capture in the brief vignette all of the communication and group interaction that developed. With the faculty member's support, students helped each other understand, interpret, criticize, and appreciate the models as they were created. Analyzing a model, understanding how it represents and misrepresents reality, and exercising caution in interpreting from the model are valuable experiences which students will draw upon after formal schooling.

Increased and Decreased Attention

Several areas are to receive increased attention in programs for liberal arts students. Students should be exposed to mathematical ideas that are new to them. The mathematics must be useful, meaningful, and not simply preparation for a higher-level course. Because liberal arts students will encounter mathematics in a variety of settings, the approach taken should involve applications from several disciplines. Students should participate in mathematical modeling, either in developing models or in evaluating how well given models fit reality. Increased attention should also be given to having students interpret real data using statistical techniques.

Traditional mathematics offerings for liberal arts students have included a wide variety of topics covered at a superficial level. It is recommended that a few topics be selected based on the needs and interests of the students. These topics should be studied in sufficient depth so that students gain insight into mathematics as a discipline and learn how to learn mathematics.

Just as the "writing across the curriculum movement" addresses the need for students to write frequently in order to improve as verbal thinkers, a "mathematics across the curriculum movement" is needed so that students develop as mathematical thinkers.

Summary

All students, including those majoring in liberal arts disciplines, should be confident in their ability to do mathematics. The mathematics that they study should prepare them to contribute in the community, to perform effectively in the workplace, and to function as independent learners in mathematics-related areas. It is particularly important that liberal arts students understand the impact that mathematics has had on art, history, literature, and many areas of human endeavor. Mathematics faculty should work closely with colleagues who teach introductory courses in the natural and social sciences, economics, and other disciplines that rely on mathematics to encourage them to reinforce and amplify the mathematics capabilities of students. Just as the "writing across the curriculum movement" addresses the need for students to write frequently in order to improve as verbal thinkers, a "mathematics across the curriculum movement" is needed so that students develop as mathematical thinkers.

"Although I don't feel that I am afraid of math as much as I am frustrated with it, I must be one of those students who suffers from math anxiety. In seventh grade I was tracked into the lower group of my math class. This was the first time that I realized that I must be having trouble in math or at least wasn't as skilled in this subject as I was in my other classes. The fact that I was put into this lower class, combined with a truly boring and monotonous teacher, simply turned me off to the subject."

This statement, written by a student during the first week of a content course for prospective elementary school teachers, expresses the feelings of many current and prospective elementary school teachers who approach their mathematics courses with trepidation and a lack of confidence. Teachers are vital to our society's cultural, technological, and economic vigor; yet, currently, we do little to promote self-confidence in learning mathematics, mathematical thinking, or deep understanding of mathematics. These are the challenges for introductory college mathematics. Our nation cannot afford to continue to have students who feel defeated by mathematics in our schools.

"The NCTM *Standards* describe the mathematics classroom as a mathematical community, where students and the teacher are actively involved in creating their learning experience. This learning community needs the strength of a knowledgeable and compassionate leader who considers the needs and talents of the student-citizens, while providing a vision of where the community is headed and support for getting there. Giving students responsibility for their own learning doesn't mean abdicating leadership. [It means] giving up some control and creating a new kind of classroom leadership that truly guides, encourages, and enlightens."

Cathy L. Seeley,
Education Consultant and former
Director of Mathematics at the Texas Education
Agency

As pointed out earlier, prospective secondary school teachers are included among the mathematics-intensive majors. The recommendations made in this section on content and pedagogy, however, apply to the mathematics education of all prospective teachers of precollege mathematics who start their collegiate mathematics education at the introductory level.

Among the recommendations in *Moving Beyond Myths* (NRC, 1991) the following are particularly relevant to the mathematics education of preservice teachers:

- Engage mathematics faculty in issues of teaching and learning.
- Teach in a way that engages students.
- Ensure sufficient numbers of school and college teachers.
- Link colleges and universities to school mathematics (p. 45).

Faculty need to seek new ways to contribute to the preparation of teachers at all levels in partnership with colleges of education and K–12 teachers of mathematics. Mathematics faculty should participate in professional discussions about mathematical preparation of future teachers. Two-year college faculty and faculty teaching introductory mathematics courses at four-year colleges and universities must become full partners in the vital task of preparing school teachers.

The Content

The mathematics studied by preservice teachers must help them develop an understanding of the subject that goes beyond what they will be expected to teach. Research on teacher education [see Brown and Borko (1992) for a summary]

indicates that the depth and character of teachers' subject matter knowledge influences both the style and substance of instruction. For example, Steinberg, Haymore, and Marks (1985) found that greater knowledge enabled teachers to convey the nature of mathematics, connect mathematical topics, provide conceptual explanations, and see problem solving as central to mathematics instruction.

In addition, classroom teachers must be able to foster the intellectual growth of their students and inculcate mathematical ways of thinking. *A Call for Change: Recommendations for the Mathematical Preparation of Teachers of Mathematics* (Leitzel, 1991) recommends that prospective teachers should learn to

- view mathematics as a system of interrelated principles;
- communicate mathematics accurately, both orally and in writing;
- understand the elements of mathematical modeling;
- use calculators and computers appropriately in the teaching and learning of mathematics; and
- appreciate the historical and cultural development of mathematics.

Prospective teachers need to recognize the relationship between what they teach and what is taught at other levels of school mathematics. According to *A Call for Change*,

> They need, for example, to understand the close parallel among the development of integer arithmetic in the elementary grades, the algebra of polynomials in the middle and early high school curriculum, and the ideas of number systems explored later in high school. They should see that counting processes and the concepts of functions and relations permeate all aspects of mathematics. They should explore the relationships between geometry and algebra and the use of one to investigate the other. (p. 3)

"All faculty must examine the extent to which their preservice mathematics education courses place subject matter in a context that is meaningful to prospective teachers and model the variety of teaching practices and assessment strategies outlined in this and other reform documents."

Task Force

Special mathematics courses for prospective elementary school teachers should revisit school mathematics topics in ways that develop deeper understanding of these topics and of the relationships among them. Furthermore, all preservice teachers should acquire a broad background in the liberal arts and sciences so that they understand how to apply mathematics in a variety of disciplines. They should also learn about the historical and current contributions of non-European cultures to mathematics and related fields; resources on this topic are available in Zaslavsky (1994) and Van Sertima (1989), as well as through the NCTM and the MAA's program SUMMA (Strengthening Underrepresented Minority Mathematics Achievement). Future teachers need to be prepared to help students who are members of groups underrepresented in mathematics to see the subject as part of their cultures.

The content standards presented in Chapter 2 provide the essential ingredients for the introductory level mathematics curriculum for K–12 teachers. Specific recommendations for the various levels of mathematical knowledge needed by teachers in grades K–4, 5–8, and 9–12 are in the *Professional Standards for Teaching Mathematics* (NCTM, 1991, pp. 135–40) and in *A Call for Change* (Leitzel, 1991). The recommendations are designed to ensure that teachers at all

grade levels have a thorough understanding of the mathematics they are teaching and a clear vision of where that mathematics is leading. Both documents recommend that

- teachers of grades K–4 study a minimum of 9 semester hours of college mathematics. Such courses assume a prerequisite of three years of high school mathematics for college-intending students or an equivalent preparation.

- teachers of grades 5–8 study a minimum of 15 semester hours of college mathematics. Such courses assume a prerequisite of four years of high school mathematics for college-intending students or an equivalent preparation.

- teachers of grades 9–12 have the equivalent of a major in mathematics. Coursework should include an integration of applications from a variety of disciplines. In addition, emphasis on problem solving and the history of mathematics is essential. Such courses assume a prerequisite of four years of high school mathematics for college-intending students or the equivalent.

Introductory college mathematics courses come at a critical stage in the development of future teachers, offering them an opportunity to move beyond their school experiences with mathematics to take a wider view of the subject. In this way, such courses can make an important contribution to K–12 mathematics education reform.

"Even though students who plan to teach do as well as or better than other mathematics majors, persistent rumor has it that the poorest students go into teaching, a rumor sustained by the unwillingness of many departments to provide rigorous courses specially designed to meet the needs of prospective teachers. If such courses are perceived as 'pushovers,' the students in them will be seen as inferior. Working partnerships between mathematics and education departments are essential if the mathematics preparation of teachers is to be improved. This is an important function, increasingly well filled by university professors of mathematics education."

Nel Noddings, in *Handbook of Research on Mathematics Teaching and Learning*, 1992, p. 210.

The Pedagogy

The report *Moving Beyond Myths* (NRC, 1991) states one of the central pedagogical problems in the training of future teachers:

> It is rare to find mathematics courses that pay equal attention to strong mathematical content, innovative curricular materials, and awareness of what research reveals about how children learn mathematics. Unless college and university mathematicians model through their teaching effective strategies that engage students in their own learning, school teachers will continue to present mathematics as a dry subject to be learned by imitation and memorization. (pp. 28–29)

If teachers are to make problem solving central to learning mathematics, they must take risks. They need to feel confident in their knowledge of mathematics, be willing to explore new mathematical ideas, and be able to stimulate active discourse in the classroom. Faculty who teach mathematics courses for prospective teachers must nurture this spirit of active inquiry. In addition, since it is common for teachers to teach the way that they were taught, faculty must use in their own classes the instructional techniques that prospective teachers will be expected to use.

Faculty should also

- follow new professional recommendations on teaching strategies;
- gain a better understanding of the mathematical needs of future elementary, middle, and high school teachers;
- rethink their teaching to promote in-depth understanding as well as a broad vision of the "big ideas" of K–12 mathematics; and
- keep abreast of the research on how students learn mathematics and adjust their teaching accordingly.

The tremendous impact of technology on education will continue to grow. Prospective teachers must understand its power and its limitations. They must know how to employ technology to enhance conceptual understanding and that technology itself should not be the main focus of instruction. Appropriate and effective uses of technology must be integrated into mathematics courses for preservice teachers.

Summary

Courses for prospective teachers should allow them to build on what they know while developing the habits of mind used by mathematicians and scientists. Students are not passive receivers of information who regurgitate correct answers on demand. The view that "learning occurs not by recording information but by interpreting it" (Resnick, 1989, p. 2) implies the need for a new vision of what it means to teach mathematics. The emphasis should shift from teaching isolated knowledge and skills to helping students apply knowledge and develop in-depth understanding of central ideas. This shift needs to occur not only in school mathematics, but also in introductory college mathematics courses taken by preservice teachers.

"There is no other decision that a teacher makes that has a greater impact on students' opportunity to learn and on their perceptions about what mathematics is than the selection or creation of the tasks with which the teacher engages the students in studying mathematics. Here the teacher is the architect, the designer of the curriculum."

Glenda Lappan and
Sarah Theule-Lubienski in
Selected Lectures from the 7th International Congress on Mathematics Education
(Quebec, August 17-23, 1992), p. 250.

CHAPTER 4:
Implications

Faculty need to experience, as learners, these curricular and pedagogical changes and then have the opportunity to model the instructional and assessment strategies for teaching the recommended content prior to incorporating them into their own instruction.

BEST COPY AVAILABLE

Although the standards for introductory college mathematics presented in Chapter 2 focus on curriculum and pedagogy, they have wide implications for institutions and their mathematics programs. This chapter presents a brief summary of recommendations on faculty development, advising and placement of students, instructional facilities and technology, assessment of students, program evaluation, and articulation among colleges and with high schools.

Faculty Development and Departmental Considerations

The standards set forth in this report mandate a complete restructuring of content and instructional strategies. While the standards emphasize meaningful mathematics and active student participation in the learning process, the college education of most current faculty emphasized algorithmic procedures and lecturing. Faculty need to experience, as learners, these curricular and pedagogical changes and then have an opportunity to model the instructional and assessment strategies for teaching the recommended content prior to incorporating them into their own instruction. Professional development opportunities must be made available to every faculty member—full-time, adjunct, and teaching assistant.

In addition, institutions should provide regular, sustained release time and other resources to support the professional development of faculty. As stated in *Moving Beyond Myths* (NRC, 1991), faculty need to "think as deeply about how to teach as about what to teach" (p. 34). Resources must be provided so that faculty can establish ongoing professional development programs on campus to meet their special needs, as well as attend off-campus professional meetings, seminars, workshops, and courses so that they can interact with colleagues from other institutions.

The instructional process recommended in this document must be designed and implemented by knowledgeable, caring, effective faculty. According to *Guidelines for the Academic Preparation of Mathematics Faculty at Two-Year Colleges* (AMATYC, 1992), effective teachers are reflective, creative, and resourceful. They use a variety of instructional methods and respond to the needs of the particular students they are teaching. Furthermore, they model behaviors they wish their students to exhibit and treat their students and colleagues in a caring and helpful manner. Faculty must provide careful academic advice, be flexible about ways in which students can meet course requirements, and simultaneously provide support to and demand commitment from their students.

While it is not the purpose of this document to offer a complete set of guidelines for the operation of mathematics departments, faculty should become familiar with and implement the recommendations of the AMATYC documents, *Guidelines for the Academic Preparation of Mathematics Faculty at Two-Year Colleges* (AMATYC, 1992) and *Guidelines for Mathematics Departments at Two-Year Colleges* (AMATYC, 1993), and the MAA's *Guidelines for Programs and Departments in Undergraduate Mathematical Sciences* (MAA, 1993). Some conclusions of these documents are

"A craft is a collection of learned skills accompanied by experienced judgement. The great advantage of thinking of teaching as a craft is the recognition that anyone can learn it. Competent teaching requires no special gift, no actor's personality, no divine spark. And if anyone can learn to be a competent teacher, then all who are employed to teach have the obligation to learn."

David S. Moore. From his remarks on receiving the MAA's 1994 Award for Distinguished College or University Teaching of Mathematics, San Francisco, January 1995. Printed in *Focus*, April 1995.

- Faculty members should be aware of and use advances in mathematics content and educational methods.

- Those who teach mathematics either in mathematics departments or in departments of developmental studies at two-year colleges should have a minimum of a master's degree that includes at least 18 semester hours of graduate work in mathematics.

- Classes must be held in a suitable environment. They must be a reasonable size (a maximum of 30 with fewer in foundation classes) to enhance the opportunity for the use of interactive learning strategies.

- Classrooms should be equipped so that computer instructional material and calculator outputs can be displayed. Computer laboratories should be available for student use.

- Adequate support services outside of class must be made available to students. Support services should include faculty who are available in their offices on a regular basis to help students, learning centers that offer professional and peer tutoring, and technology specialists who can help students in computer laboratories.

- The administrator of the department should be aware of the standards, promote new ideas and experimentation, and facilitate the establishment of appropriate learning environments. Opportunities should be provided for faculty to discuss anticipated changes.

"As a general perspective, I feel too many mathematics teachers act as if they are teaching students who are what they think they were at age 18. One catch is that few, if any, of us really remember what we were like 20 or 30 years ago. Further, we lose sight of the fact that 98% of the students around us at that time never went on to careers in mathematics or science, nor should we expect that much more than 2% of our current students will. . . . We must recognize that we are addressing a much broader audience who is not motivated by what may have motivated us and we must adapt our offerings accordingly."

Sheldon Gordon,
Suffolk Community College

A major concern in two- and four-year colleges and universities is the access of students in introductory college mathematics courses to full-time faculty. A sizable segment of this curriculum is taught by graduate students at universities and by adjuncts at two- and four-year colleges. An excessive dependence on instructional personnel who may not contribute to curriculum development, advising, and other department responsibilities and who may not be available to provide students with help outside of the classroom can place undo hardship on a department and on the students.

Both AMATYC (1993) and MAA (1993) recommend that adjunct staffing should be kept at a minimum. When used, such faculty should have the same qualifications as full-time faculty. They should participate in professional development activities, they should be kept fully informed of departmental activities and policies, and they should be included in departmental activities whenever possible. The MAA (1993) recommends that graduate teaching assistants be closely supervised by full-time faculty.

"Research into how students learn mandates the changes in pedagogy. The demands of the real world mandate the curricular changes."

Task Force

The limited use of adjunct faculty can enhance mathematics education in some circumstances. Adjunct faculty whose full-time employment involves applying mathematics can bring special expertise and breadth to the department's offerings. In addition, adjunct faculty who are regularly employed in other area institutions bring a fresh perspective to the department and may provide an interface between local high schools and postsecondary institutions.

Faculty who have already begun to incorporate the proposed standards into

their teaching report that some students feel uncomfortable with the deviation from "traditional" instruction. Even students who have, by and large, been unsuccessful learning in a traditional classroom resist change. Faculty must be prepared to smooth the transition as much as possible by

- providing students with detailed instructions to clarify faculty expectations of them,
- forming student study groups and resource centers specifically designed to help students understand the changes being instituted,
- listening to student concerns and frustrations expressed orally or in writing and following their suggestions where advisable,
- providing students with evidence that the changes will enhance their learning, and
- offering students alternatives, with traditional instructional strategies occasionally being the alternative.

"Students were assigned mathematics laboratory experiments for which they wrote laboratory reports. This type of work was new to them and they indicated great frustration with the assignments. The professor revised his course procedures to include very specific instructions for writing such reports and included a sample laboratory report from a previous semester. Difficulties for later students were diminished."

Task Force

In addition, faculty need the support of a learning center as a source of tutoring and other forms of student assistance, especially for times when faculty are not available. Learning centers should also provide a setting for students to work in groups. In this regard, learning center staff must also be knowledgeable of the standards for introductory college mathematics so that instruction received in these centers is in concert with classroom procedures.

"Revitalizing and reforming undergraduate mathematics education is one of the principal challenges facing the profession today" says the report *Recognition and Rewards in the Mathematical Sciences* (Moore, 1994). One of the report's guiding principles states: "Each department should ensure that contributions to teaching and related activities and to service are among the primary factors of importance in the recognition and reward system" (p. 30). It is critical that institutions reward faculty for their active involvement in curricular and pedagogical reform.

Advising and Placement

"Departments should have established policies and procedures for placement into introductory mathematical science courses. It is important that these policies be well understood and disseminated across the institution" (MAA, 1993, p. 9). Placement testing should be applied equitably to all students and must be statistically valid so that students have the opportunity to be successful, as well as challenged in their coursework. Implementation of the standards advocated in this document and the reform taking place at the high school level make it imperative that new placement tests, consistent with these initiatives, be developed and continually updated as further revisions in curriculum and pedagogy occur and new technologies are used. Placement tests should include questions that do more than merely test the students' mastery of algorithmic skills. They should provide measurements of students' abilities to think critically, use technology,

"The goal of any placement program is to place students into courses in which they will be successful and challenged."

John Jenkins, Embry-Riddle Aeronautical University

solve problems, and communicate about mathematics using a variety of methods.

Placement procedures should include more than review of students' high school records, college entrance examinations, and appropriate mathematics placement tests. Students should have the opportunity to meet with faculty to discuss how their proposed placement relates to their educational, career, and personal goals. Final decisions on the proper placement of students in mathematics must be made by the mathematics department in conjunction with the student.

Students should also receive general academic advising. Faculty act as advisors at some colleges, while on other campuses special counselors do all the advising. When faculty are involved in advising, they should receive training and appropriate compensation for these duties.

Advising is generally associated with such activities as helping students select courses and checking to be sure that degree requirements are satisfied. However, changes in the collegiate student population—more returning adults, more academically underprepared students, more who are the first in their families to attend college, and so forth—necessitate increased emphasis on advising. In addition to the traditional duties of an advisor, the following matters should be considered:

- identifying students with learning disabilities and using the assistance of relevant campus agencies,
- disseminating information about career opportunities involving mathematics,
- providing students interested in mathematics with the opportunity to meet in an informal setting with faculty and other students with the same interest,
- providing enrichment activities for mathematics students, and
- addressing the learning needs of a culturally diverse student population.

A mathematics laboratory should provide students with activities designed to guide them in the construction of their own understanding of mathematical concepts, strengthen their ability to critically apply mathematics, and encourage the use of a variety of skills and content to solve nontraditional problems.

Laboratory and Learning Center Facilities

This document advocates the use of mathematics laboratories in the teaching of mathematics. A mathematics laboratory should provide students with activities designed to guide them in the construction of their own understanding of mathematical concepts, strengthen their ability to critically apply mathematics, and encourage the use of a variety of skills and content to solve nontraditional problems. Students should work in groups, use cooperative learning strategies, and communicate what they have learned either orally or in writing. A variety of tools and techniques should be used in the laboratory, including faculty demonstrations and hands-on student work with manipulatives, data gathering instruments, graphing calculators, and computers.

In addition to calculators, computers, and other technological tools, an effective laboratory environment includes helpful instructional staff members and movable furniture to facilitate group activities. Colleges must budget adequate funds to hire and train needed staff and to purchase and maintain appropriate equipment as well as allocate the necessary amount of space for mathematics

laboratories. The addition of a laboratory component should not be used as an excuse to add more topics to the curriculum. Rather, it should be used to enhance understanding of the topics faculty have decided are important for students to learn.

Mathematics programs should also include a learning center facility, staffed by professional and peer tutors, where students can gather to get extra help, work on assignments, and socialize. In addition, the facility can serve as a home base for a mathematics club, a library, and an area for informal faculty and student interaction.

TECHNOLOGY

Technology is changing and will continue to change the way mathematics is done around the world. Almost every research study on the use of calculators or computers in the classroom reports that "the performance of groups using calculators equaled or exceeded that of control groups denied calculator use" (MSEB, 1990, p. 22). For example, Dunham and Dick (1994) reviewed research on the classroom use of graphing calculators:

> The early reports from research indicate that graphing calculators have the potential dramatically to affect teaching and learning mathematics, particularly in the fundamental areas of functions and graphs. Graphing calculators can empower students to be better problem solvers. Graphing calculators can facilitate changes in students' and teachers' classroom roles, resulting in more interactive and exploratory learning environments. (p. 444)

Computers, graphing calculators, educational television, computer-based telecommunications, video discs, and other technological tools and related software should be fully utilized in college classrooms [see Dunham (1993), Dunham and Dick (1994), Foley (1990), and Leitzel (1991)].

Although most students can afford to purchase their own graphing calculators, colleges should establish lease or loan programs for needy students. In addition, adequate funds must be allocated by colleges to purchase other technological resources.

Technology that enhances learning is available, and students will use it whenever they realize its power, regardless of whether professors allow it in their classrooms. Mathematics faculty must adapt to this reality and help students use technology appropriately so that they can be competitive in the workforce and adequately prepared for future study.

ASSESSMENT OF STUDENT OUTCOMES

Assessment must be viewed as an integral part of instruction. Mathematics faculty must use assessment strategies that not only determine the extent of student learning but also support student learning. Furthermore, the assessment instruments must measure the full range of what students are expected to learn, not just what is easy to measure. Assessment should build excellence into the educational process by providing regular feedback to students and faculty about learning and instruction.

"With technology–
Some mathematics becomes more important.
Some mathematics becomes less important.
Some mathematics becomes possible."

Henry Pollak (Retired),
Bell Laboratories

"The assessment procedures should do justice to the goals of the curriculum and to the students–context independent generalized testing is unjust when for instruction the context includes the real world of mathematics itself, at least in the realistic mathematics education approach. An essential question is: 'Does assessment reflect the theory of instruction and learning?' "

Jan de Lange in
Selected Lectures from the 7th International Congress on Mathematics Education
(Quebec, August 17-23, 1992), p. 235.

"Organizing my portfolio gave me a wonderful opportunity to evaluate my progress. When I compare where I am now with where I was, mathematically, at the semester's beginning, I am very proud of myself! My confidence in my ability has continued to grow throughout the semester, and I've actually reached the point where I enjoy the challenges presented to me instead of becoming instantaneously defeated."

A student's comments

As curriculum standards, instructional strategies, and student outcomes change, effective standards for assessment and accountability must follow. A new national understanding of assessment must be built upon the fundamental principles presented in such publications as *Measuring Up* (MSEB, 1993), *For Good Measure* (MSEB, 1991), *Assessment Standards for School Mathematics* (NCTM, 1995), and *Assessment of Student Learning for Improving the Undergraduate Major in Mathematics* (Madison, 1995).

Expanded views of the more traditional forms of assessment are needed. For example, a variety of forms of testing should be used—essay, short answer, open-ended, and multiple-choice questions, and oral as well as written exercises and problems. Furthermore, group testing is particularly appropriate for students who are accustomed to working in groups during regular course activities. Students should also be expected to write essays, do boardwork, do group projects and laboratory reports, and make oral presentations. Assessment instruments should measure not only students' knowledge of mathematics content, but also their ability to solve problems, to communicate, to work in groups, and to read technical material.

Students should use technology on virtually all tests just as they do in regular coursework. At the same time, students should be held accountable for learning certain basic skills. One assessment technique involves identifying a competency set of basic skills to be mastered. Students must achieve competency (say, 90%) on a "gateway" test (taken without calculators or computers) or they do not pass the course, regardless of their performance on other assessments. They may retake the test at given intervals until they pass. This reinforces the message that certain skills are mandatory and essential. (See Megginson, 1994 for a description of and research on the gateway testing program at the University of Michigan.)

Portfolios can be used to make general assessments of student learning. A student's mathematics portfolio is similar to an artist's portfolio, containing representative samples of all assessment instruments used in class. These might include quizzes, examinations, homework, laboratory and other written reports, copies of group projects, as well as notes or videotapes from oral presentations and teacher-student interviews.

Just as many of the recommended assessment techniques will be new to some faculty, they will be new to some students. Whatever mix of assessment instruments are used, it is of paramount importance that students be fully informed about the manner in which performance will be assessed.

All activities that are part of the learning experience should be a part of the assessment of that learning. Good assessment practices should be indistinguishable from good instructional practices. The assessment instrument or method is not the end product of learning but rather part of the learning process.

The intent of any program evaluation should be to make recommendations for improvement and updating while retaining the effective aspects of the program.

PROGRAM EVALUATION

The effectiveness of any mathematics program depends on its ongoing revision and revitalization based on regular evaluations. The intent of any program evaluation should be to make recommendations for improvement and updating while retaining the effective aspects of the program.

Although mathematics faculty should carry the primary responsibility for program evaluation, comprehensive information about the mathematics program

should be collected from all areas affected— faculty, students, administrators, and community or employer advisors. Whoever directs the evaluation should collect information that identifies strengths, weaknesses, and recommendations for improvement. In addition, the evaluation should include periodic discussions with graduates of programs and their employers to determine the effectiveness of these programs in meeting the needs of business, industry, and the community. Nearby high school teachers and mathematics faculty from surrounding colleges and universities should also be surveyed. Such coordination of instruction across institutions will foster better articulation from one program to another and ensure more consistent mathematical experiences for the students.

The standards for intellectual development, for content, and for pedagogy contained in this document should serve as criteria for evaluation of the instructional aspects of programs. Furthermore, a review of faculty and support staff, physical facilities, equipment, and supporting materials is important to determine whether they are appropriate to and sufficient for the objectives of the program. Program evaluation should also include review and evaluation of assessment methods.

Articulation with High Schools, with Other Colleges and Universities, and with Employers

The formation of local consortia of high schools, colleges, and universities enables mathematics faculty at all levels to work in concert to improve mathematics education.

Educational reform must not be done in isolation. The formation of local consortia of high schools, colleges, and universities enables mathematics faculty at all levels to work in concert to improve mathematics education. Representatives from industry, business, and government who are the future employers of college graduates should be included as full partners in educational reform. The practitioners in the field and from client disciplines can provide vital up-to-date input on mathematical methods and problem solving strategies that they use. Such cooperation provides continuity in the educational experiences of students from high school to postsecondary institutions and between such institutions with better uniformity and consistency both in content and approach. Secondary and postsecondary institutions should work together to assure students entering each level of mathematics in postsecondary education that their preparation is appropriate for that level.

One reform project, the Maricopa Mathematics Consortium (M^2C), has built a working consortium of five local high school districts, the ten Maricopa County colleges, and Arizona State University. The consortium will develop and implement a complete curriculum for introductory college mathematics based on the theme "mathematics in context." They are developing partnerships with other disciplines, as well as with local business and industry, to clarify the contexts in which people use mathematics to understand the world better. The collaboration between educational levels serves to advance curricular change at each level.

Crossroads in Mathematics takes a strong stand on what mathematics is important for students to learn, thereby raising important questions about articulation. Will students who participate in an introductory college mathematics program that is guided by the standards have difficulty if they transfer into a traditional mathematics program? Will the lack of emphasis on symbol manipulation skills impede students? While it is too soon for extensive research to be available on current reform initiatives, there is evidence to suggest that

transfer problems should not be a major obstacle.

For example, F.S. Gordon (1995) compared the algebraic skills of students using the NSF-funded Math Modeling/PreCalculus Reform Project course materials with those of students using traditional college algebra and trigonometry materials. The group using the reform materials had higher mean scores on six of seven common questions that appeared on final examinations. The differences in means on three of the questions were significant at the 5% level in favor of the reform group. Gordon comments in her paper that the "real" difference in the classes was brought home to her when she was answering homework questions in both classes on one particular day. The reform class raised a question that involved using an exponential function to model population growth in California. The traditional class asked about factoring $(x + 1)^4 - 9$.

Tidmore (1994) compared the performance of students using materials produced by the Calculus Consortium based at Harvard University (CCH) with that of students using a traditional text on a common Calculus I final examination at Baylor University. The CCH materials emphasize educational principles analogous to the standards presented in this document. The results indicated that CCH students had a significantly higher overall performance, and they were higher at each ability level. An analysis of scores on problems from identifiable areas of study indicated that the CCH group did better on problems that were graphical, numerical, or conceptual in nature. On problems that were algebraic or traditional in nature, the scores were very similar.

Students can take reformed mathematics courses without major loss of computational abilities and they are not handicapped when they study higher levels of mathematics.

Bookman (1994) found that, compared to a group of students taught traditional calculus, students in Project CALC at Duke University performed better in many ways: They became better problem solvers, they were observed to be more actively engaged in the study of calculus, and they had higher continuation rates from Calculus I to Calculus II (and from there to more advanced mathematics classes). In addition, there were virtually no differences in their grades in mathematically oriented courses taken after Calculus I. On the other hand, the Project CALC students did less well on computational skills involving symbolic manipulation. The faculty involved felt this problem could be remedied by including more practice on routine calculations.

While these studies do not directly answer the articulation questions, they do point out that students can take reformed mathematics courses without major loss of computational abilities, and they are not handicapped when they study higher levels of mathematics. Furthermore, if students understand mathematical principles and practice independent learning, they should be able to function in a traditional setting.

Communication among educational institutions, between high schools and colleges, and between mathematics and other departments will ease the transition from a traditional to a reformed approach to mathematics education. Documents like this one help to accelerate the pace of reform and look forward to the day when there is no need to ask about articulation problems.

SUMMARY

This document recommends a complete restructuring both in content and pedagogy of introductory college mathematics. Changes in the structure of course offerings and in the content of the mathematics curriculum influence all areas of

instruction. But these changes will flounder without proper support of the faculty who are expected to carry them out. If faculty are given the opportunities and the tools they need to make substantive contributions to the reform effort, they will become effective advocates of the changes that are needed in such areas as advising and placement, facilities, technology, assessment, program evaluation, and articulation.

CHAPTER 5:
Implementation

Adoption and implementation of the standards will require a systemic, nationwide effort.

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Crossroads in Mathematics: Standards for Introductory College Mathematics Before Calculus provides a framework for the development of improved curriculum and pedagogy. Adoption and implementation of the standards will require a systemic, nationwide effort.

Faculty, with strong support from administrators, are primarily responsible for implementing educational reform. Professional mathematics organizations must lead the effort to promote this implementation by making their support highly visible at their meetings and in their publications and by providing professional development opportunities. Furthermore, business, industry, governing bodies, accrediting agencies, and public and private funding agencies must unite behind the efforts to reform higher education. Their voices are heard and respected in the local and national communities.

This chapter describes an integrated national plan of local, state, and national actions that will ensure that reform takes place in a lasting and pervasive way.

Institutional Recommendations

Mathematics departments must become campus leaders in improving all classroom instruction. Faculty can begin the process of reform by discussing and evaluating their own local curriculum and pedagogy using the standards proposed in this document as benchmarks. They should then become active participants in applying the standards to develop methods for improving instruction in their own institutions. The following ideas can be used to spark departmental discussion:

- The introductory mathematics curriculum must become less cumbersome and more meaningful. Topics that have become outmoded because of readily available technology should be deleted. Content should be integrated so that topics naturally connect and build on each other.
- Improvements in pedagogy are necessary for the student to become a more active, involved learner.
- Learning and problem solving through teamwork in the mathematics classroom must reflect the team approach to problem solving and communication expected in the world of work.
- Technology must be used whenever appropriate so that the content is understandable and useful to students.
- Mathematics class must be a fertile ground for exploration and experimentation.

College administrators must recognize the need for curricular and pedagogical improvements. They must provide leadership, support, and incentives for human resource development as well as the necessary space and technology to implement the standards. Such changes will involve adjustments to course descriptions, credit hours, scheduling, and fees.

"The document is designed to attract national attention to the issue of implementing reform in introductory college mathematics and thereby provide support to mathematics faculty and impetus for them to become involved."

Task Force

Professional Organizations

All national mathematics professional organizations and their related affiliated groups are encouraged to participate in implementing the standards for improving instruction in introductory college mathematics. AMATYC, AMS, MAA, NADE, NCTM, and SIAM should assume a coordinated leadership role. Their newsletters and journals, as well as sessions and workshops at national and regional meetings, can play a major role in promoting the standards. Professional organizations should also take the lead in forming networks of key leaders in two- and four-year colleges and universities and in encouraging members to implement the standards (NSF, 1992).

In addition, national support and coordinating organizations, such as the National Association of State Science and Mathematics Coalitions (NASSMC); Mathematicians for Education Reform (MER); the Mathematical Sciences Education Board (MSEB); the Conference Board of the Mathematical Sciences (CBMS); and the Coordinating Board of AMATYC, MAA, and NCTM should provide endorsement and assistance in the dissemination and implementation of the standards outlined in this document.

"Researchers in mathematics education have recognized that students' beliefs, faculty beliefs, and beliefs of groups outside the classroom are factors to be considered in educational reform. . . . Classroom change will only take effect if students will accommodate and approve of these changes and the community outside the classroom will be supportive of a virtual revolution in the educational process with all its inherent consequences."

Pao-sheng Hsu.
UME Trends, March 1995, p. 16.

State mathematics organizations should also assume a leadership role in the initiation, discussion, and development of improved curriculum and pedagogy in introductory college mathematics. Where more than one state mathematics professional organization exists, they should coordinate their efforts. To support these efforts, state governing and coordinating boards should become knowledgeable advocates for reform; funding agencies should provide for the necessary infrastructure (NSF, 1991).

At a local level, the organization of consortia of two- and four-year colleges and universities will be encouraged through various state and regional mathematics groups. These consortia should provide a framework for development and dissemination of materials, professional development activities, and institutional research on instructional effectiveness.

Other Recommendations

Regional accreditation associations should become advocates for reform in introductory college mathematics. This step is critical in alerting administrators to the need for change.

The report *Matching Actions and Challenges* (NSF, 1991) sets a goal that, by 1996, 25 percent of the nation's faculty who teach introductory college mathematics should be supporting educational reform at the introductory level. The National Science Foundation, other governmental bodies, and private foundations, should support this goal.

Proposed Regional Workshops

A series of regional workshops, each bringing together faculty members and administrators, will develop implementation plans for the standards at the regional, college, and classroom levels. Workshops should have sufficient numbers of attendees to ensure that working groups will continue to direct and coordinate reform initiatives after the workshops are over.

These workshops and other implementation efforts will

- inform a wide audience of the reform issues and ideas for improving curriculum and instruction,
- review current exemplary materials and activities and share information and insights about current reform projects in introductory college mathematics,
- set goals and plan for continuing the reform efforts,
- develop assessment strategies appropriate to the standards,
- empower teams of faculty and consortia of two- and four-year colleges and universities to work jointly toward reform, and
- enhance the ability of two- and four-year college and university mathematics faculty to obtain funding for reform projects.

"Major reforms will not happen without dealing with teacher anxieties and frustrations as they attempt to change."

Task Force

The workshops should lay the groundwork for systemic change and improvement in introductory college mathematics.

Development of Materials

Development of new materials based on the standards set forth in this document is essential to lasting reform. Publishing companies and manufacturers of calculators and software can help to stimulate the development of new materials. Professional organizations and mathematics faculty must educate commercial suppliers about the nature of the standards. Publishers, in turn, can support faculty in their efforts toward reform by making available quality instructional materials. Additionally, curriculum project writing teams, composed of a consortium of two- and four-year colleges and universities, may jointly develop and pilot new materials in their own classrooms. To ensure consistency with the NCTM Standards, writing teams should involve high school mathematics faculty as consultants.

"I used to believe that mathematics was just learning the procedures and memorizing the rules and if you could do that you would be able to do mathematics. After this semester, I see mathematics not just as problems, but as a special way to look at things."

A student's comments

Summary

Implementation of these standards as a new vision of introductory college mathematics will require a concerted national effort. National and regional mathematics organizations and two- and four-year colleges and universities must cooperate to provide curricular and faculty development. Resources provided by funding agencies will be critical catalysts in reform.

CHAPTER 6:
Looking to the Future

The standards are intended as a call to action, a catalyst for national discussion, and an inspiration for creative thinking and innovation.

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Think back to the opening vignette, which described the uncertainties of a two-year college faculty member teaching an experimental section of algebra: Although excited by the material and encouraged by the students' responses, she worried that the course did not include the skill development that conventional courses offer. Having read this report, would you respond to her concerns differently now? How would your response differ?

We hope that the standards presented in this document inspire new ways of looking at introductory college mathematics. Too often a domain of low expectations and unfulfilled potential, introductory college mathematics courses hold the promise of opening new paths to future learning and fulfilling careers to an often neglected segment of the student population. The standards are intended as a call to action, a catalyst for national discussion, and an inspiration for creative thinking and innovation. We hope that this document convinces you to devote some time to thinking about how to provide better instruction and an improved curriculum to benefit those for whom we all desire the best—our students.

Today, introductory college mathematics plays a critical role in so many professions that improving instruction at this level is essential for our nation's vitality. The NCTM *Standards* (NCTM, 1989) set forth an agenda for change in mathematics education at the K-12 levels; numerous calculus reform projects are pioneering new ideas at the calculus level. These reforms make it imperative that higher education take a fresh look at introductory college mathematics and how it connects to other levels of study.

Educational reform is an evolving process, and this document offers guidelines for an initial phase. As the recommendations presented here are tested and evaluated, new recommendations and new ideas will emerge. While the implementation period will likely be chaotic and uncertain, it will certainly be exciting, challenging, and professionally fulfilling. We believe this standards-based reform effort will provide all students with a more engaging and valuable learning experience. Our students deserve no less; our nation requires no less; and we must demand no less of ourselves.

We believe this standards-based reform effort will provide all students with a more engaging and valuable learning experience. Our students deserve no less; our nation requires no less; and we must demand no less of ourselves.

APPENDIX: ILLUSTRATIVE EXAMPLES

PROBLEM SOLVING

> Most visitors to the program thought that the heart of our project was group learning. . . . But the real core was the problem sets which drove the group interaction. One of the greatest challenges that we faced and still face today was figuring out suitable mathematical tasks for students that not only would help them to crystallize their emerging understanding . . . , but that also would show them the beauty of the subject. (Treisman, 1992, p. 368)

Problems provide opportunities for students to learn and do mathematics. The mathematical tasks faculty present to students shape what and how they learn and are critical in assessment of their learning. Within a traditional instructional sequence, problems are typically given to students after they have learned some piece of mathematics as a way to apply what they have learned. However, problems can also be used as a catalyst for learning. They can be the driving force leading to student discovery and invention. In a course driven by problems, "mathematics is what you have left over after you have invented ways to solve a problem and reflected on those inventions" (A. Selden and J. Selden, 1994, p. 5, paraphrasing Robert B. Davis).

This appendix presents a variety of examples of problems aimed at capturing the spirit and vision of the standards in this document. The problems are arranged in order of increasing difficulty and complexity. Most offer story lines. The level of realism varies, with some problems using genuine data.

A variety of content strands are represented, with an emphasis on material that traditionally has not been highlighted in introductory college mathematics. Brief remarks address the nature and role of each problem. The solutions given or suggested often span several levels of mathematical sophistication. Hence, many of the problems could be used in a variety of courses or within a single class in which students' backgrounds vary significantly. Such problems should help the reader see how the same standards can and should apply to all mathematics taught in college below the level of calculus.

The problems assume students have access to modern technology. Calculator and computer technology is becoming increasingly ubiquitous, portable, affordable, and user-friendly. This technology gives students access to certain mathematical ideas and problems at an earlier stage in the students' development than is possible without technology. Technology opens the door to exploration and experimentation because many examples can be investigated in a brief span of time. Learning mathematics empirically by discovery, with carefully sequenced tasks and the guidance of knowledgeable faculty, is a much more viable instructional option with technology than without.

The problems can accommodate a variety of teaching and learning styles. However, it is generally assumed that the mathematics classroom is a place where students are "engaged in collaborative, mathematical practice—sometimes working with others in overt ways, and always working collaboratively with peers and with the teacher in a sense of shared community and shared norms for the practice of mathematical thinking and reasoning" (Silver, 1994, p. 316).

Many of the problems can best be solved by students working in small groups. Some may be used as classroom or homework exercises. The more extensive problems may be used as laboratory projects. To get students to organize their thoughts and express them in writing, individual or group reports can be required. Students will generally write better reports if they are given detailed directions. Here is a sample set of directions for a group laboratory report:

> **All of the work on this lab should be done collaboratively. You will be asked to grade your own effort as well as the efforts of the other members of your group. Within the lab report clearly indicate the primary author for each section. Authorship should be shared fairly among the group members. Each lab report should have three parts: (a) Begin with a paragraph of introduction giving an overview of the nature and purposes of the lab in your own words. (b) The introduction should be followed by a write-up of each activity—what you did, how you did it, and what conclusions you reached. Include tables and graphs as appropriate. Show your work and indicate your thought processes in an organized fashion. (c) Each report should end with a paragraph of summary, conclusions, and reflections.**

PRODUCING, GATHERING, AND USING GENUINE DATA

Problem solving can be made more meaningful to students if they are given opportunities to produce, gather, and use data. Gathering and producing data and all of the related issues form an important area of statistics. One need not go into all of these issues in detail, however, to use such data throughout introductory college mathematics. It is best to use *genuine data*, that is, real data from real sources. These can be gathered by students through computer and calculator gathering devices, library research, and surveys.

Modern technology makes obtaining real-time data much easier. This can be done by linking probes to computers or graphing calculators via special data collection devices that are readily available. Probes, or sensors, can measure temperature, light intensity, voltage, motion, sound, acidity, and other variables of scientific interest. These data can be instantly transferred into the memory of a computer or graphing calculator and used in data analysis and modeling.

Library research and surveys are also excellent ways to obtain real data. Newspapers, magazines, and simple 15- to 20-item class surveys that include such variables as height, shoe size, and gender offer a rich source of authentic data for statistical analysis. Surveys can build a sense of class unity and identity. They also offer faculty the opportunity to introduce students to the ideas of random samples and bias in data.

THE EXAMPLES

1. Chicken Chunks (Organizing Data and Pattern Recognition)

The following problem can be solved by generating and organizing relevant data and looking for patterns. The problem is challenging but does not require sophisticated mathematics.

Problem. In an effort to keep up with the expanded menu of competitors, Hal's Hamburger Haven now sells "chicken chunks" in boxes of 6, 9, and 20. Notice that some amounts of chunks cannot be purchased; for example, you cannot buy 8 or 14 chunks. Why? What is the greatest number of chunks that you *cannot* buy at Hal's?

2. Midpoint on a Number Line (Formula Development)

The following problem should be given in advance of the midpoint formula so that students can invent the formula. It becomes less abstract, contextually richer, and more accessible if students work in small groups using a physical number line (e.g., a meter stick or a yard stick) in the solution process. Freudenthal (1991, pp. 36-37) discusses typical student behaviors, ways to guide students' reinvention of the formula, and possible extensions for this problem type.

Problem. Locate each of the following midpoints:

a. Locate the midpoint between 18 cm and 76 cm on a meter stick. Describe the mathematics used to help calculate the midpoint.
b. Locate the midpoint between 15 cm and 18 cm on a meter stick. Describe the mathematics used to help calculate the midpoint.
c. Locate the midpoint between 5 1/2 in. and 24 3/4 in. on a yard stick. Describe the mathematics used to help calculate the midpoint.
d. Locate the midpoint between the numbers x and y. Describe the procedure used to determine the midpoint expression.

Extension. This problem can be extended to having students develop an algebraic expression for a point that is any fractional distance from x to y.

3. Tennis Ball Can (Measurement and Geometry Formula)

"Clearly" the height of a can of three tennis balls is greater than the circumference of the can. Or is that really the case? The following problem can be used to compare results obtained by measurement to results obtained by using a geometric formula. Groups of students should be given a common tennis ball can along with two paper tapes, one calibrated in inches (to eighths or sixteenths) and the other in centimeters (to tenths).

Problem. Which is greater, the height of the cylindrical tennis ball can or its circumference? By how much?

a. Estimate the answer by simple observation.
b. Measure the height and the diameter of the can. Determine the circumference using a geometric formula. Compare the height to the circumference.
c. Measure the circumference of the can with a paper tape measure. How does the measurement compare with the results obtained from the geometric formula? Compare the height to the circumference.

Notice that the problem does not indicate which paper tape to use for the measurements and the level of precision needed in the answers. At the conclusion of the exercise, faculty should lead students to discuss the merits of the U.S. Customary System and the metric system. Furthermore, faculty may also use the example to discuss rounding and the potential errors in the answers. The ideas of precision, accuracy, and rules for operations with numbers that are the results of measurements play a key role in the education of science and technical majors.

4. Proportional Relationships

Students should know how to use the idea of proportions as a basic problem solving strategy.

Problem. Decide if a proportional relationship is present in each of the following situations. If so, describe the relationship and use it to solve the problem.

a. In one small office within a corporation, 6 out of 9 management positions are held by women. If that rate holds throughout the corporation, how many of the 180 management positions are held by women?
b. One inch is the same length as 2.54 cm. About how many inches is 75 cm?
c. If the length of a side of a triangle is 3 in. and its area is 9 sq. in., what is the length of the corresponding side of a similar triangle whose area is 36 sq. in.?
d. The scale on a roadmap is 1 inch = 20 miles. How many miles are there between two cities 11.5 inches apart on a map?

5. Projecting Weekly Wages (Linear Models)

At the introductory stages of modeling, students should be expected to examine the mathematics behind a proposed model. For example, if they have reason to believe there are equal increases in y for equal increases in x, they should try a linear model. If they reason that increases are in a geometric progression, they should try an exponential model. This problem involves the development of a linear model at a basic level of mathematical sophistication and introduces students to the idea of curve fitting.

Problem. Shonda is interested in a job as a server at the Restaurante Ricardo to earn money while in college. She was told that servers are paid $12 per day for working the dinner shift on weekends plus tips. She was advised that servers averaged $4.50 per table served. Below is Shonda's analysis of her projected weekend (three-day) wages before taxes:

tables served	0	5	10	15	20	25	30
weekend wages	$36	$58.50	$81	$103.50	$126	$148.50	$171

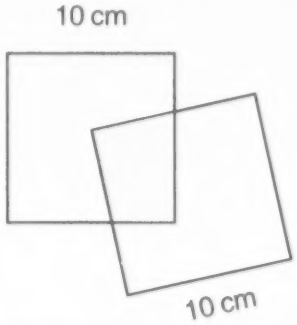
a. Extend Shonda's analysis by making the table show her wages when 7, 13, 35, and 40 tables are served.
b. Make a graphical presentation of the extended table from part a.
c. Are the wages increasing at a constant rate? (That is, are the data linear?)
d. Determine an algebraic model that expresses Shonda's projected wages as a function of the number of tables that she serves one weekend.
e. Graph the function on the same set of axes used in part b.
f. Estimate from the graph in part e Shonda's projected wages for a weekend that she serves 40 tables. Now determine the answer algebraically.
g. How many tables must Shonda serve in order to project weekend wages of $200?
h. Do you think that it is possible for Shonda to earn $700 in one weekend? Carefully explain.

Notice that this problem involves using different scales on each axis. The scale on the vertical axis involves relatively large numbers. The writing assignment in part h gives students the opportunity to consider the range of function values for which the function has meaning.

6. A Puzzling Problem (Geometry and Deductive Proof)

While this document calls for the use of problems that are real-world applications, all problems do not have to be of that type. Problems that are easy to state, easy for students to understand, and that have an unusual twist can pique student interest. The following problem has a nonintuitive answer. Note that the proof that is requested is needed to validate conjectures that are made. In this sense it is an integral part of the solution process, and thus, enhances the mathematics involved.

Problem. The figure below shows two congruent overlapping 10 cm by 10 cm squares. The tilted square is movable, but one vertex always remains at the center of the other square that does not move.



a. What is the largest possible area of the overlapping shaded region?
b. Validate your answer to part a by proving that your answer is correct.

Extension. In parts a and b the students should conjecture and then prove that the area of the shaded region remains a constant 25 cm^2. Does the perimeter of the shaded region also remain constant? If so, find the constant perimeter. If not, determine the minimum and maximum perimeters.

Using Manipulatives. Colored, transparent 10 cm by 10 cm plastic squares are commercially available. If students can experiment with, say, a red and a blue square, they are likely to gain insight into the problem.

Using Technology. The squares in this problem can be "constructed" using a computer and commercially available interactive dynamic geometry software. Once constructed, the overlapping polygonal region can be specified and its area and perimeter can be measured. The software will give a quasi-continuous readout of these two measurements as one square is rotated about the center of the other square. This will enable the students to support previously made conjectures which would then be verified using traditional methods. The software can also be used to draw auxillary line segments to aid in the construction of a proof.

7. Population Growth Comparison (Exponential Growth)

Students should have the opportunity to solve problems that lend themselves to the use of multiple strategies.

Problem. The population of Huntsville, Texas, was 25,854 in 1985 and is increasing by 1.55% each year. The population of Conroe, Texas, was 22,314 in 1985 and is increasing by 4.35% each year.

a. Make a table showing the changing population of the two cities.
b. In which year did or will Conroe's population first exceed Huntsville's?

Note: According to their respective Chambers of Commerce, the population of Huntsville was 23,936 in 1980 and 27,925 in 1990 and the population of Conroe was 18,034 in 1980 and 27,610 in 1990. This implies average annual growth rates of 1.55% and 4.35%, respectively, which can be used for forward or backward population projections from the given years. The 1985 populations in the stated problem are the geometric means of the census figures. This problem can be readily adapted to your locality.

The problem can be solved using (1) arithmetic, by building a table one line at a time; (2) recursion, on a calculator (see Figure 1); (3) a sequence defined by a recursive formula; (4) a sequence with an explicit nonrecursive formula; and by functions (5) numerically using tables (see Figure 2), (6) graphically using the ZOOM-IN feature of a graphing utility, (7) using the SOLVE feature of a calculator or computer software, or (8) the traditional way using logarithms.

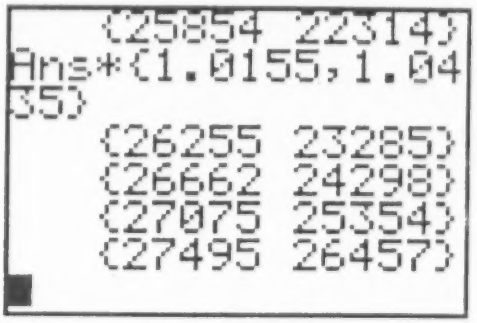


Figure 1. Recursion on a large-screen calculator.

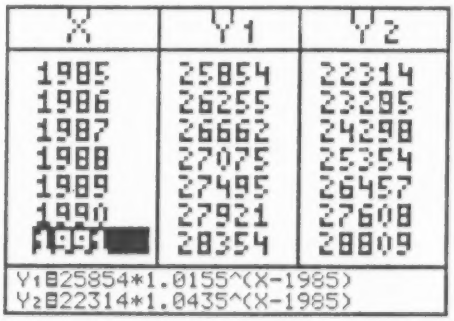


Figure 2. A table of values generated by algebraic formulas.

8. Problem Posing

It is a good idea to let students create their own problems from a given context once they have had some experience in problem solving. This gives students a great deal of insight into the problem-solving process. It is empowering for students to realize they can create problems. Theirs will be tougher than common textbook problems in many cases. When given the following problem situation, students often generate problems involving inequalities or questions about domain and range of functions.

Problem situation. Janet is making a rectangular end table for her living room. She has decided the tabletop should have a surface area of 625 sq. in.

Task. Write three different types of questions that could be asked about this problem situation. Solve two of them.

Some students responses:

- What perimeters are possible?
- What would the perimeter be if Janet makes it a square?
- The distance between the end of the sofa and the wall where the end table will go is only 18 in. How long could the end table be?
- If the table is at least 20 in. wide, how long could it be?
- What range of dimensions (length and width) are reasonable?
- What are the dimensions if the table is twice as long as it is wide?

Table 1 suggests an ongoing solution strategy for the last question that requires no algebra. Problem posing can be done early in an introductory college mathematics curriculum; it is not an activity appropriate for only the precalculus level. The method of successive approximations, or systematic guess-and-check, used in Table 1 is one that can be used by students at any level. Note the "Process" columns. They are valuable because, as is often the case, the *arithmetic process* can give insight into the *algebraic structure.* The "Process" columns can be used as an instructional bridge to algebra.

Table 1. Areas for tables of various dimensions, if the table is twice as long as it is wide

Length	Width (in.)		Area (in.^{2})	
(in.)	Process	Result	Process	Result
40	40/2	20	40 x 20	800
30	30/2	15	30 x 15	450
34	34/2	17	34 x 20	578
36	36/2	18	36 x 20	648
35	35/2	17.5	35 x 17.5	612.5
35.4	35.4/2	17.7	35.4 x 17.7	626.58
35.3	35.3/2	----	----	----

9. Maintaining Pool Chlorine Level (Numerical Experimentation)

This problem presents students with a genuine problem situation for which they must make and test conjectures about its solution. With the support of technology, students can experiment to determine a solution to the problem.

Problem. Chlorine is used to control microorganisms in the water of a pool. Too much chlorine produces burning eyes; too little, and slime develops. Here are some facts about pool care:

- Chlorine dissipates in reaction to bacteria and to the sun at a rate of about 15% of the amount present per day.
- The optimal concentration of chlorine in a pool is from 1 to 2 parts per million (ppm), although it is safe to swim when the concentration is as high as 3 ppm.
- It is normal practice to add small amounts of chlorine every day to maintain a concentration within the 1 to 2 ppm ideal.

a. Use a calculator to find the amount of chlorine (in ppm) remaining each day for 10 days, if the level at time zero is 3 ppm and no more is added.
b. Graph the concentration of chlorine (in ppm), c, as a function of time, t, for the data determined in part a. Find the interval of time over which the chlorine level is optimal for humans.
c. If chlorine is added every day, another model is necessary. Use a computer or calculator spreadsheet to model this system for a 21-day period when the concentration is 3 ppm at time zero.

i. Try adding 1 ppm each day. Clearly that is too much, but does the pool water turn to chlorine? What is the largest amount of chlorine attainable?
ii. Try adding 0.1 ppm everyday. Does this process yield ideal conditions in the long run?
iii. Find a daily dosage that stabilizes the concentration of chlorine at 1.5 ppm.

In solving this problem, students must understand thoroughly the parameters of the situation—the facts about pool care. Through experimentation afforded by technology, students can design and determine an effective solution to the chlorine problem and judge whether their solution is reasonable.

10. Basketball Performance Factors (Matrix Operations)

Students should be able to use matrices to help them solve a wide variety of mathematics problems. The following elementary problem may be used to introduce students to the use of matrices to organize data and to the use of matrix operations for solving problems.

Problem. The women's basketball team at the University of Connecticut completed a perfect 35-0 season by defeating the University of Tennessee to win the 1995 NCAA women's basketball championship. Senior Rebecca Lobo led her team in scoring with 17 points. Does the fact that she contributed 17 points indicate the total worth of her contribution to the victory? Does the fact that Nykesha Sales scored only 10 points indicate that she made much less of a contribution? Obviously, other performance factors contribute to a team's effort. Positive factors include assists (a), steals (s), rebounds (r), and blocked shots (b). On the other hand, turnovers (t) and personal fouls (f) are examples of negative factors. Here is a listing of some of the nonshooting performance factors for the seven Connecticut players who participated in the championship game.

	a	s	r	b	t	f
Elliott	3	1	7	0	5	3
Lobo	2	0	8	2	2	4
Wolters	0	0	3	2	1	4
Rizzotti	3	3	3	0	4	3
Webber	2	0	1	0	0	1
Sales	3	3	6	0	1	3
Berube	2	0	3	0	3	0

a. Let each instance of a positive factor count 1 point and each instance of a negative factor count -1 point. Find each player's nonshooting performance score.
b. Now think of the box score as a 7 x 6 matrix A and, similarly, the associated points as a 6 x 1 matrix B:

$$A = \begin{bmatrix} 3 & 1 & 7 & 0 & 5 & 3 \\ 2 & 0 & 8 & 2 & 2 & 4 \\ 0 & 0 & 3 & 2 & 1 & 4 \\ 3 & 3 & 3 & 0 & 4 & 3 \\ 2 & 0 & 1 & 0 & 0 & 1 \\ 3 & 3 & 6 & 0 & 1 & 3 \\ 2 & 0 & 3 & 0 & 3 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

Use your calculator or computer to find the product AB. How does the 7 x 1 product matrix compare with the calculations made in part a?

c. Based on your observations here, explain how matrix multiplication works. Try your ideas out by multiplying the following on your calculator and comparing it to your hand or mental calculations. Make a statement about the relative dimensions of the factor matrices and the size of the resulting product.

i. $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 4 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

ii. $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 4 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}$

iii. $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}$

Explain any "strangeness" you observe here.

Generalizations and Extensions

d. The owner of the New York Yankees baseball team has developed a uniform incentive clause for hitters on his team. He has decided to pay them \$100 for each time at bat, \$400 for each run scored, \$300 for each hit, and \$500 for each run batted in. Find the sports section in a newspaper and calculate the value of the incentive clause for one game for the first six hitters in the Yankees' lineup. If it is not baseball season, go to the library and find the results for an old game. Set up two matrices, the 6 x 4 performance matrix and the 4 x 1 incentive-clause matrix, so that your calculator can do the multiplication for you.

e. In a series of games, you can calculate the value of the incentive clause for each game and then add them together. As an alternative, explain how the calculator could be used to add the performance statistics matrices of all of the games before multiplying the incentive matrix. In other words, explain how matrix addition should work.

f. Compare and contrast matrix addition with real number addition. Also, compare and contrast matrix multiplication with real number multiplication. Include a discussion of various properties such as associativity, commutativity, existence of identities and inverses, and the distributive property.

11. Dart Board Problem (Geometry and Probability)

This problem combines the ideas of area and probability in a meaningful manner.

Problem. A circular dart board has a single dot right in the middle. For each dart that hits the board 5 points are awarded. If a dart lands closer to the center dot than the outer edge, an additional 10 points are awarded. If a dart is thrown at random so as to hit the board, what is the probability that the extra 10 points will be awarded?

This problem can be solved using simple geometry and illustrates that doubling all linear dimensions of a plane figure quadruples its area. Alternatively, a computer or calculator program based on a random number generator can be used to simulate the dart throwing, and the relative frequencies obtained can be used to estimate the theoretical probability. This so-called "Monte Carlo" simulation can be used to illustrate the relationships between the means of several trials obtained by individuals, groups, and the entire class—leading to a discussion of weighted means and the central limit theorem.

It is not being suggested that teachers should have students write the needed computer or calculator program. Teachers

could supply students with the needed program, have interested students write the program, do a classroom demonstration, or have each student find a few distances without programming and then combine the individual frequencies.

Extension. If the dart board is square, the problem becomes more challenging. Using traditional methods, it would be a fairly difficult calculus problem, but it can be solved by the Monte Carlo method with no use of calculus (see Figure 3).

Figure 3. A simulation of 100 darts randomly hitting a square dart board.

12. Analyzing Planet Data (Modeling and Re-expression of Data)

A model developed for genuine data should be tested for both goodness of fit and a lack of pattern in the residuals. Mathematical models are seldom exact, and the imperfections are of serious concern. The residuals—the differences between actual and predicted values—measure the imperfections of a model. A good model should have residuals that are relatively small and randomly distributed. A correlation coefficient close to ± 1 indicates errors that are small in size but gives no indication of the nature of the distribution of residuals as the following example indicates.

The following problem leads to Kepler's third law, which states that the square of the period of orbit for each planet is equal to the cube of its semimajor axis, provided the earth's period and semimajor axis are used as the units of time and distance. It links data analysis and algebra and involves students in an interesting use of logarithms. This problem works well as an activity for groups of three or four students using cooperative learning. The solution presented below assumes this arrangement.

Problem. Consider the planet data given in Table 2. According to Kepler's first law, the orbit of each planet is an ellipse with the sun at one focus. Let T be the orbit period (i.e., the time required for one full revolution around the sun), and let a be the length of the orbit's semimajor axis.

A. Plot T versus a. Determine an algebraic model that is appropriate for the data, and justify your choice of model. Be sure to consider residuals in your justification.
B. Plot $\log T$ versus $\log a$. Determine and justify a model for this new plot.
C. Compare the numerical, graphical, and algebraic aspects of your two models. How are the models related? Why?
D. Revise your model in part A using the earth's period and semimajor axis as the units of time and distance. State the resulting model in words.
E. Predict Pluto's orbit period given that its semimajor axis is 5.9 billion km.

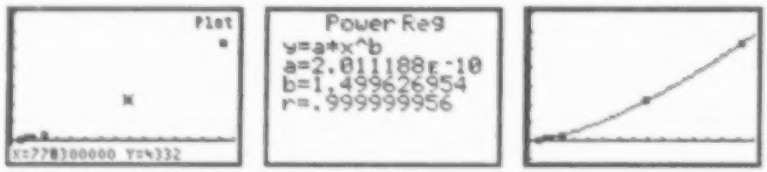
Table 2. Orbit Periods and Semimajor Axes for Six Planets

Planet	Period of Revolution (days)	Semimajor Axis (km)
Mercury	88	57,900,000
Venus	225	108,200,000
Earth	365.2	149,600,000
Mars	687	227,900,000
Jupiter	4332	778,300,000
Saturn	10,760	1,427,000,000

Source: Shupe, Dorr, Payne, Hunsiker et al., 1992, plate 116.

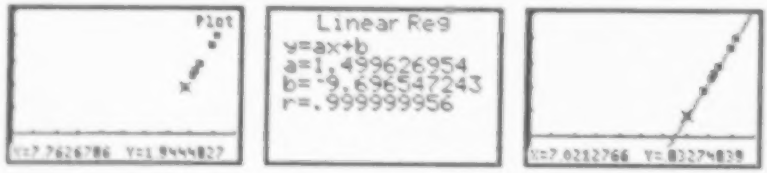
Part A. Because each x coordinate represents a semimajor axis, the points corresponding to the four inner planets are closely clustered, just as the planets themselves are closely clustered in the solar system. The scatterplot suggests that the relationship is nonlinear. Nevertheless, many groups of students will be tempted by the correlation coefficient of $r = 0.9924$ to conclude the relationship is linear. The least squares line, however, predicts Mercury to have a period of revolution of -363 days and yields a residual plot with a clearly nonrandom pattern.

Students will likely try several other options before deciding that a power function provides the best model, if they reach that conclusion at all. The example shows that mathematical and scientific considerations are essential parts of the modeling process. Students should attempt to use scientific information at the beginning of a modeling process to suggest potential models, or at the end of the process to validate models that they develop. In this case, a power function model with a power of 1.5 yields a nearly perfect correlation of 1, suggesting the data points lie very close to the graph of the power function.



A power of 1.5 is consistent with Kepler's third law. Students can overlay the graph of the regression equation model over the scatter plot to further confirm the choice of models.

Part B. Technology makes it easy for students to obtain the log-log data and its plot. Many students will be surprised at how different this plot is from the previous plot. The points are spaced more uniformly. The relationship appears to be linear. So, students will readily pick a linear regression model. It is hoped, most groups will notice how much easier it is to find this model and that logarithms greatly simplify the relationship between the variables.



Part C. Students should notice that the correlation coefficients match in the models in parts A and B, and the slope of the line in B matches the power in A. To explain these relationships, students will need to sort through the conflicting

notation. The variables a, b, x, and y have all been used in multiple ways. The situation with a is the worst. It has been used as a regression coefficient in each model and also represents the semimajor axis. The algebra reveals that the two models are equivalent. It also suggests a connection between the models that most students would not have anticipated, that the logarithm of the coefficient in A is the constant term, or y intercept, in B. Students should be encouraged to support this with numerical evidence.

Part D. Students can change the periods to earth-based time units (i.e., years) by dividing the list of periods by 365.2, the number of days per year. Students need to change the distances to earth-based units, too. This can be done by dividing the list of distances by its earth entry. The earth-based distances obtained are in *astronomical units*. The power regression option will yield a model that is essentially $T = a^{3/2}$, which is equivalent to Kepler's equation $T^2 = a^3$.

Part E. This may seem anticlimactic because students merely need to substitute Pluto's semimajor axis value into the model in part A. The predictive power of scientific theory, however, should not be discounted. As an illustration, gravitational theory led Lowell to postulate the existence and general location of Pluto 25 years before it was first sighted. Library research is a natural follow-up to this planet problem. For example, students could be asked to check whether the answer found for part E is consistent with information in the literature, to verify that the data for Uranus and Neptune fit the model, or to write a report on Kepler's laws.

13. A Pure Mathematics Problem (Absolute Value Functions)

Thus far, most of the example problems have had a context, a story line. Some of the problems are obviously contrived (e.g., the dart board problem, 12), while others are based on genuine data (e.g., the planet problem, 13). Mathematics presented in context is often more attractive to students and will engage their interest and attention to a greater degree than mathematics without a context.

Lest there be any doubt, however, there are many worthwhile and important pure mathematics problems, like the one presented below. This problem presents pure mathematics in a way that allows for group interaction and use of technology to develop critical concepts.

Part A. Sketch a graph for each function. On the same axes, in a different color, dash in the graph produced by omitting the absolute value operation. Check your graphs with a graphing utility.

1. $g(x) = |x|^2$
2. $h(x) = |x|^3$
3. $k(x) = \dfrac{1}{|x|}$
4. $t(x) = \sqrt{|x|}$
5. $m(x) = e^{|x|}$
6. $l(x) = \ln|x|$

Conclusions: Describe how applying the absolute value operation before the characteristic operation affects the graph.

Part B. Sketch a graph for each function. On the same axes, in a different color, dash in the graph produced by omitting the absolute value operation. Check your graphs with a graphing utility.

1. $G(x) = |x^2|$
2. $H(x) = |x^3|$
3. $K(x) = \left|\dfrac{1}{x}\right|$
4. $T(x) = \sqrt{|x|}$
5. $M(x) = |e^x|$
6. $L(x) = |\ln x|$

Conclusions: Describe how applying the absolute value operation after the characteristic operation affects the graph.

Part C. Sketch a graph for each function. On the same axes, in a different color, dash in the graph produced by omitting the absolute value operation.

1. $f_1(x) = |x^2 - 4|$
2. $f_2(x) = |x^2 - 5x + 6|$
3. $f_3(x) = 2^{|x|} + 3$
4. $f_4(x) = 2^{|x|} - 4$

Conclusions: Do your descriptions from part A and part B explain how to produce the graphs involving absolute value from their standard graphs? Give steps to produce such a graph.

Part D. Generalize your conclusions. Use the steps you listed in part C to sketch the graph of each of the following functions. Don't use any other method. Now by checking points or using your grapher determine if the graph is correct.

1. $g_1(x) = \dfrac{1}{|x + 1|}$
2. $g_2(x) = \dfrac{1}{|x - 3|}$
3. $g_3(x) = \ln|x + 1|$
4. $g_4(x) = \ln|x - 3|$

Conclusions: If your steps did not cover these cases, explain why. Redo your descriptions to cover these more general cases.

14. A Writing Assignment

These standards emphasize the role of writing in the learning of mathematics. Students can learn about how mathematics is used in their major fields of study by interviewing practitioners or doing library research and then writing reports. For example, business students might report on how finance charges are determined on credit card purchases.

In a very practical sense, technical majors will be expected to write reports for their employers. Consider the following problem:

Problem. The production plans for a factory indicate the need for 3600 Model 12 units for the upcoming year (June through May). The demand for the Model 12 is even throughout the year. Each Model 12 unit requires a prefabricated encasement that is obtained from an outside vendor. The vendor will allow the factory to determine the number of shipments in which the 3,600 encasements will be sent but mandates that an equal number of encasements be sent in each shipment. The charges are as follows:

- Each encasement costs $7.26 but a 6% price increase has been planned for December 1.
- A service charge of $50 is due on each shipment.

Additionally, the factory is only able to store 600 encasements at any one time. Provide the vender with a report indicating the number of shipments to make and when they should be received by the factory. Provide your employer with justification for your decision.

15. A Laboratory Project—Fluid Flow (Data Gathering and Curve Fitting)

Mathematics at the introductory level should be taught as a laboratory science. Stevens (1993) presents several examples of sources of data that lend themselves to laboratory modeling projects. They include the heights of corn seedlings as a function of time, normal high temperature as a function of the day of the year, and the height of a burning candle as a function of the length of time that it has been burning. Students might also attempt to model the price of one brand of laundry detergent as a function of the weight or volume of the detergent, or develop a population model for a country like Mexico over the past 30 years.

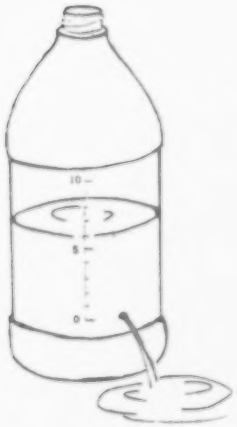
In some situations, scientific principles or previous experiences indicate the type of model that should be developed. In other cases, students have to experiment with a variety of models to determine the one that fits best. Once the data have been gathered and graphed, students can determine the parameters of the hypothesized model algebraically, through trial and error (seeing how well the proposed model fits the points), or by using regression programs on a calculator or computer.

The following project does not need any special data gathering equipment other than a stopwatch. It is assumed that students are working in groups.

Introduction. As liquid flows through a hole at the bottom of a container with a constant cross-sectional area, the height of the liquid above the hole is a function of the time that the liquid has been flowing. The fact that the function is a quadratic function is an application of Torricelli's law.

Objective. Students will gather data and fit the data to a quadratic function using educated trial and error. They will determine approximations of the needed parameters algebraically and then refine their approximations graphically.

Procedures



1. Calibrate the height of a clear plastic container that has a constant cross-sectional area in centimeters. Drill a small hole at zero. It is best to put the zero point a few centimeters up from the bottom of the container. (The whole container does not have to have a constant cross-sectional area. Simply calibrate the portion that is constant.) A height of about 10 cm is sufficient. Fill the container with colored water to the 10-cm mark while holding your finger over the hole.
2. Release your finger from the hole and at the same time have a groupmate start the stopwatch. Determine the length of time that it takes to pass the 8-cm, 6-cm, and 4-cm marks and record the results. You might want to do the experiment three times and use the mean time at each level.
3. You must now determine a quadratic function

$$h = at^2 + bt + c$$

that best fits the data. Plot your four data points [(0,10) is the first point]. Use algebra to determine a quadratic function through three of the points. Modify the parameters of that function so that it "best" fits all four of the data points. Here, the best fit is simply determined by looking at the graph.

4. Once you have determined a function that closely fits all four data points, predict when the fluid will be at $h = 0$. Fill the container to a height of 10 cm and measure the time that it takes the water level to get to zero. How accurate was your prediction?

Conclusion. Write a report that explains the procedures that you used and provide a description of your results. Explain the reasoning that you used. Specifically explain how changing the parameters affected the shape of your graph.

Possible Model. A one gallon window washer fluid container was used to gather the following data.

t (sec)	0	31	68	107
h (cm)	10	8	6	4

Figure 4 represents the graph of $h = 0.0001t^2 - 0.07t + 10$. Improvements can be made to the fit by changing the coefficients of t^2 *and* t.

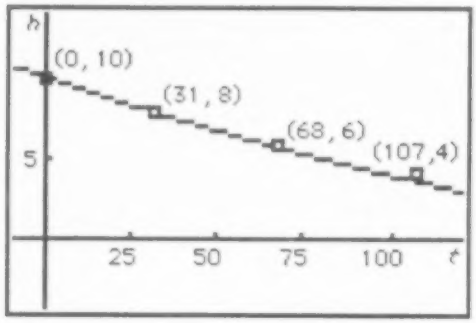


Figure 4. The graph of $h = 0.0001t^2 - 0.07t + 10$.

Closing Remarks

In this appendix, problem ideas have been presented to illustrate how the standards of this document can be realized in the college mathematics classroom. It is hoped that the problems will stimulate the imagination of college mathematics faculty to develop rich and meaningful activities for their students that will help them learn mathematics and appreciate its power and utility.

There are many sources for interesting problems and curricular materials. The National Science Foundation supports projects in undergraduate mathematics course and curriculum development. Professional journals, such as *The AMATYC Review*, the *College Mathematics Journal*, the *Mathematics Teacher*, and *PRIMUS* offer many classroom ideas and interesting problems. The Consortium for Mathematics and Its Applications (COMAP) has developed numerous applications modules which are available for purchase. Professional meetings and workshops offer many valuable ideas and invaluable collegial interaction.

Professional organizations and commercial publishers have already developed some textbooks, software, and other materials in keeping with the spirit of this document. As time goes on, the number and quality of such materials will likely increase.

REFERENCES

Albers, D. J., Loftsgaarden, D. O., Rung, D. C., & Watkins, A. E. (1992). *Statistical abstracts of undergraduate programs in the mathematical sciences and computer science in the United States, 1990-91 CBMS Survey* (MAA Notes Number 23). Washington, DC: Mathematical Association of America.

American Mathematical Association of Two-Year Colleges. (1992). *Guidelines for the academic preparation of mathematics faculty at two-year colleges* (Report of the Qualifications Subcommittee of the Education Committee). Memphis, TN: Author. (Reprinted in *The AMATYC Review, 15*(1), 40-49.)

American Mathematical Association of Two-Year Colleges. (1993). *Guidelines for mathematics departments at two-year colleges* (Report of the Two-Year College Mathematics Departments Subcommittee of the Education Committee). Memphis, TN: Author. (Reprinted in *AMATYC NEWS, 9*(1), centerpiece.)

Becker, J. R., & Pence, B. J. (1994). The teaching and learning of college mathematics: Current status and future directions. In J. J. Kaput & E. Dubinsky (Eds.), *Research issues in undergraduate mathematics learning* (pp. 5-14). Washington, DC: Mathematical Association of America.

Bookman, J. (1994, June). *Final Report: Evaluation of Project CALC 1989 - 1993 (National Science Foundation grant # DUE - 8953961).* Paper presented at the Third Conference on the Teaching of Calculus, Ann Arbor, MI.

Brophy, J., & Good, T. L. (1986). Teacher behavior and student achievement. In M. C. Whitrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 328-375). New York: Macmillan.

Brown, C. A., & Borko, H. (1992). Becoming a mathematics teacher. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 209-239). New York: Macmillan.

Burrill, G., Choate, J., Phillips, E., & Westegaard, S. (Conference Organizers). (1993). *Algebra for the twenty-first century: Proceedings of the August 1992 conference.* Reston, VA: National Council of Teachers of Mathematics.

Choppin, J. M. (1994). Spiral through recursion. *Mathematics Teacher, 87*, 504-508.

Collins, T. W., Gentry, D. K., & Crawley, V. O. (Co-Chairs). (1993). *Gaining the competitive edge: Critical issues in science and engineering technician education* (A report from a workshop sponsored by the National Science Foundation and the Federal Coordinating Council for Science, Engineering, and Technology). Washington, DC: National Science Foundation.

Crocker, D. A. (1990). What has happened to calculus reform? *The AMATYC Review, 12*(1), 62-66.

Crocker, D. A. (1991). Constructivism and mathematics education. *The AMATYC Review, 13*(1), 66-70.

Crocker, D. A. (1992). Cooperative learning. *The AMATYC Review, 14*(1), 78-83.

Davis, R. M. (Ed.). (1989). *A curriculum in flux: Mathematics at two-year colleges* (A report of the Joint Subcommittee on Mathematics Curriculum at Two-Year Colleges). Washington, DC: Mathematical Association of America.

Demana, F. (1990). Improving college readiness through school/university articulation. In N. Fisher, H. Keynes, & P. Wagreich (Eds.), *Mathematicians and education reform: Proceedings of the July 6-8, 1988 workshop* (pp. 131-144). Providence, RI: American Mathematical Society.

Demana, F. (1994). College precalculus courses: Challenges and change. In A. Solow (Ed.), *Preparing for a new calculus* (pp. 14-20). Washington, DC: Mathematical Association of America.

Dossey, J. A. (1991). Discrete mathematics: The math of our time. In M. J. Kenney & C. R. Hirsch (Eds.), *Discrete mathematics across the curriculum, K-12: 1991 yearbook of the National Council of Teachers of Mathematics* (pp. 1-9). Reston, VA: National Council of Teachers of Mathematics.

Druckman, D., & Bjork, R. A. (Eds.). (1994). *Learning, remembering, believing: Enhancing human performance.* Committee on Techniques for the Enhancement of Human Performance, National Research Council. Washington, DC: National Academy Press.

Dunham, P. H. (1993, May). Does using calculators work? The jury is almost in. *UME Trends*, p. 8.

Dunham, P. H., & Dick, T. P. (1994). Research on graphing calculators. *Mathematics Teacher, 87*, 440-445.

Eisenberg, T., & Dreyfus, T. (1994). On understanding how students learn to visualize function transformations. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education I,* (pp. 45-68). Providence, RI: American Mathematical Society.

Foley, G. D. (1990). Accessing advanced concepts with technology. *The AMATYC Review, 11*(2), 58-64.

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures.* Dordrecht, The Netherlands: Kluwer.

Gardiner, A. D. (1991). A cautionary note. In M. J. Kenney & C. R. Hirsch (Eds.), *Discrete mathematics across the curriculum, K-12: 1991 yearbook of the National Council of Teachers of Mathematics* (pp. 10-17). Reston, VA: National Council of Teachers of Mathematics.

Goldblatt, A. (1994, May). Undergraduate remedial mathematics: Can it be remedied? *UME Trends*, p. 7.

Goldstein, J. A. (Chair). (1989). Mathematics appreciation courses. In L. A. Steen (Ed.), *Reshaping college mathematics* (109-125). Washington, DC: Mathematical Association of America. (A reedited version of the report of the CUPM Panel on Mathematics Appreciation Courses originally published in the *American Mathematical Monthly,* Vol. 90 (1983), pp. 40-51 and the Center Section.)

Gordon, F. S. (1995). *Assessing how well precalculus students do while* Functioning in the Real World. *PRIMUS, 5,* 253-263.

Haney, B., & Testone, S. (1990). After semester workshop for developmental mathematics. *The AMATYC Review 11* (2), 54-56.

Hart, E. W. (1991). Discrete mathematics: An exciting and necessary addition to the secondary school curriculum. In M. J. Kenney & C. R. Hirsch (Eds.), *Discrete mathematics across the curriculum, K-12: 1991 yearbook of the National Council of Teachers of Mathematics* (pp. 67-77). Reston, VA: National Council of Teachers of Mathematics.

Haver, W., & Turbeville, G. (1995). An appropriate culminating mathematics course. *The AMATYC Review, 16* (2), 45-50.

Hector, J. H. (1992). Graphical insights into elementary functions. In J. T. Fey & C. R. Hirsch (Eds.), *Calculators in mathematics education* (pp. 131-137), 1992 Yearbook of the National Council of Teachers of Mathematics. Reston, VA: National Council of Teachers of Mathematics

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan.

Keith, S. Z., & Leitzel, J. R. C. (1994, May). General education mathematics: A new challenge for departments. *UME Trends,* pp. 6-7.

Leitzel, J. R. C. (Ed.). (1991). *A call for change: Recommendations for the mathematical preparation of teachers of mathematics.* Washington, DC: Mathematical Association of America.

Madison, B. L. (Chair). (1995). Assessment of student learning for improving the undergraduate major in mathematics. (Prepared by the Mathematical Association of America Subcommittee on Assessment of the Committee on the Undergraduate Programs in Mathematics). *Focus, 15* (3), 24-28.

Mathematical Association of America. (1993). *Guidelines for programs and departments in undergraduate mathematical sciences.* Washington, DC: Author.

Mathematical Sciences Education Board. (1990). *Reshaping school mathematics.* Washington, DC: National Academy Press.

Mathematical Sciences Education Board. (1991). *For good measure.* (Report of the National Summit on Mathematics Assessment). Washington, DC: National Academy Press.

Mathematical Sciences Education Board. (1993). *Measuring up.* Washington, DC: National Academy Press.

Megginson, R. E. (1994). A gateway testing program at the University of Michigan. In A. Solow (Ed.), *Preparing for a new calculus* (pp. 85-88). Washington, DC: Mathematical Association of America.

Mesterson-Gibbons, M. (1989). *A concrete approach to mathematical modeling.* Reading, MA: Addison-Wesley.

Moore, C. C. (Chair). (1994). *Recognition and rewards in the mathematical sciences.* (Report of the Joint Policy Board for Mathematics Committee on Professional Recognition and Rewards). Providence, RI: American Mathematical Society.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics.* Reston, VA: Author.

National Research Council. (1989). *Everybody counts.* Washington, DC: National Academy Press.

National Research Council. (1990). *A challenge of numbers.* Washington, DC: National Academy Press.

National Research Council. (1991). *Moving beyond myths: Revitalizing undergraduate mathematics.* Washington, DC: National Academy Press.

National Science Foundation. (1991). *Matching actions and challenges: Report of a National Science Foundation workshop on science, engineering and mathematics education in two-year colleges.* Washington, DC: Author.

National Science Foundation. (1992). *Partners in progress: Report of a National Science Foundation workshop on the role of professional societies in science, technology, engineering, and mathematics education in two-year colleges.* Washington, DC: Author.

Pollak, H. (1987, May). Presentation at the Mathematical Sciences Education Board Frameworks Conference, Minneapolis, MN.

Reich, R. P. (1993). Strategies for a changing workforce. *Educational Record, 74* (4), 22-23.

Resnick, L. B. (1989). Introduction. In L. B. Resnick (Ed.), *Knowing, learning and instruction: Essays in honor of Robert Glaser* (pp. 1-24). Hillsdale, NJ: Lawrence Erlbaum.

Resnick, L. B., & Klopfer, L. E. (1989). Toward the thinking curriculum: An overview. In L. B. Resnick & L. E. Klopfer (Eds.), *Toward the thinking curriculum: Current Cognitive Research* (pp. 1-18). (1989 Yearbook of the American Society for Curriculum Development). Washington, DC: American Society for Curriculum Development.

Ross, S. C. (Ed.). (1994, January). NSF Beat: $2.1 million in NSF calculus grants. *UME Trends,* pp. 1, 10, and 15.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Secada, W. G. (1992). Race, ethnicity, social class, language, and achievement in mathematics. In D. A. Grouws (Ed.), *Handbook on research in mathematics education* (pp. 623-660). New York: Macmillan.

Secretary's Commission on Achieving Necessary Skills. (1991). *What work requires of schools: A SCANS report for America 2000.* Washington, DC: U.S. Department of Labor.

Secretary's Commission on Achieving Necessary Skills. (1993). *Teaching the SCANS competencies.* Washington, DC: U.S. Department of Labor.

Selden, A., & Selden, J. (1994, March). Mathematics education displays breadth in Cincinnati. *UME Trends,* pp. 4-5.

Shoaf-Grubbs, M. M. (1994). The effect of the graphing calculator on female students' spatial visualization skills and level-of-understanding in elementary graphing and algebra concepts. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education, I* (pp. 169-194). Providence, RI: American Mathematical Society.

Shupe, J. F. (Project Director & Chief Cartographer), Dorr, J. F. (Art Director), Payne, O. G. A. M. (Text Ed.), Hunsiker, M. B. (Chief, Map Research) et al. (1992). *National Geographic atlas of the world* (rev. 6th ed.). Washington, DC: National Geographic Society.

Siegel, M. J. (Chair). (1989). Discrete mathematics. In L. A. Steen (Ed.), *Reshaping college mathematics* (pp. 61-84). Washington, DC: Mathematical Association of America. (A re-edited 1986 committee report of the Mathematical Association of America ad hoc Committee on Discrete Mathematics.)

Silver, E. A. (1994). Mathematics thinking and reasoning for all students: Moving from rhetoric to reality. In D. F. Robitaille, D. H. Wheeler, & C. Kieran (Eds.), *Selected lectures from the 7th International Congress on Mathematical Education* (pp. 311-326). Sainte-Foy, Québec, Canada: Les Presses de l'Université Laval.

Solow, A. (Ed.). (1994). *Preparing for a new calculus.* Washington, DC: Mathematical Association of America.

Sons, L. R. (Chair). (1995). *Quantitative reasoning for college graduates: A complement to the Standards.* (Report of the CUPM Subcommittee on Quantitative Literacy). Washington, DC: Mathematical Association of America.

Steen, L. A. (Ed.). (1989). *Reshaping college mathematics.* Washington, DC: Mathematical Association of America.

Steen, L. A. (1990). Pattern. In L. A. Steen (Ed.), *On the shoulders of giants* (pp. 1-10). Washington, DC: National Academy Press.

Steinberg, R., Haymore, J., & Marks, R. (1985). *Teachers' knowledge and structuring content in mathematics.* Paper presented at the annual meeting of the American Educational Research Association, Chicago.

Stevens, J. (1993). Generating and analyzing data. *Mathematics Teacher, 86,* 475-487.

Swetz, F. (1991). Incorporating mathematical modeling into the curriculum. *Mathematics Teacher, 84,* 358-365.

Tidmore, G. (1994, June). *A comparison of calculus materials used at Baylor University.* Paper presented at the Third Conference on the Teaching of Calculus, Ann Arbor, MI.

Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education I* (pp. 21-44). Providence, RI: American Mathematical Society.

Treisman, U. (1992). Studying students studying calculus: A look at the lives of minority mathematics students in college (a Mary P. Dolciani lecture). *College Mathematics Journal, 23,* 362-372.

Tucker, A. C., & Leitzel, J. R. C. (1994). *Assessing calculus reform efforts.* Washington, DC: Mathematical Association of America.

U.S. Department of Education. (1990, July). *National goals for education.* Washington, DC: U.S. Government Printing Office.

Van Sertima, I. (Ed.). (1989). *Blacks in science, ancient and modern.* New Brunswick, NJ: Transaction Books.

Zaslavsky, C. (1994). *Fear of math.* New Brunswick, NJ: Rutgers University Press.

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